A UNIFIED APPROACH TO STRUCTURAL LIMITS

WITH APPLICATION TO THE STUDY OF LIMITS OF GRAPHS WITH BOUNDED TREE-DEPTH

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Abstract. In this paper we introduce a general framework for the study of limits of relational structures in general and graphs in particular, which is based on model theory and analysis. We show how the various approaches to graph limits fit to this framework and that they naturally appear as "tractable cases" of a general theory. As an outcome of our theory, we provide extensions of known results and identify some new cases exhibiting specific properties suggesting that their study could be more accessible than the full general case. The second part of the paper is devoted to the study of such a case, namely limits of graphs (and structures) with bounded diameter connected components. We prove that in this case the convergence can be "almost" studied component-wise. Eventually, we consider the specific case of limits of graphs with bounded tree-depth, motivated by their role of elementary brick these graphs play in decompositions of sparse graphs, and give an explicit construction of a limit object in this case. This limit object is a graph built on a standard probability space with the property that every first-order definable set of tuples is measurable. This is an example of the general concept of *modeling* we introduce here. It is also the first "intermediate class" with explicitly defined limit structures.

Contents

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1. Introduction

To facilitate the study of the asymptotic properties of finite graphs in a sequence $G_1, G_2, \dots, G_n, \dots$, it is natural to introduce notions of *structural convergence*. By structural convergence, we mean that we are interested in the characteristics of a typical vertex (or group of vertices) in the graph G_n, as n grows to infinity. This convergence can, for instance, be expressed by the convergence of the properties' distributions. It may also be the case that they can be obtained by sampling either a measurable limit graph, or a random limit graph defined by means of edge probabilities (or densities).

While the later case corresponds in particular to *graphons* (limit objects introduced by Lovász et al. [11, 12, 46] to study extremal properties of dense graphs based on subgraph statistics), the former case corresponds in particular to *graphing* (limit objects for Benjamini-Schramm convergence of graphs with bounded degrees [6, 23], which is based of vertex neighbourhood statistics). Both models are probabilistic in that they count statistics of subgraphs (or equivalently of homomorphisms).

It seems that the existential approach typical for decision problems, structural graph theory and model theory on the one side and the counting approach typical for statistics and probabilistic approach on the other side have nothing much in common and lead to different directions: on the one side to study, say, definability of various classes and the properties of the homomorphism order and on the other side properties of partition functions. It has been repeatedly stated that these two extremes are somehow incompatible and lead to different area of study (see e.g. [10, 34]).

In this paper we take a radically different approach: We propose a model which is a mixture of the analytic, model theoretic and algebraic approach. It is also a mixture of existential and probabilistic approach: typically what we count is the probability of existential extension properties. Precisely, our approach is based on the *Stone pairing* $\langle \phi, G\rangle$ of a first-order formula ϕ (with set of free variables $\mathrm{Fv}(\phi)$) and a graph G, which is defined by following expression

$$\langle \phi, G\rangle = \frac{|\{(v_1, \dots, v_{|\mathrm{Fv}(\phi)|}) \in G^{|\mathrm{Fv}(\phi)|} : G \models \phi(v_1, \dots, v_{|\mathrm{Fv}(\phi)|})\}|}{|G|^{|\mathrm{Fv}(\phi)|}}.$$

A sequence of graphs $(G_n)_{n\in\mathbb{N}}$ is FO-*convergent* if, for every first order formula ϕ (in the language of graphs), the values $\langle \phi, G_n\rangle$ converge as $n \to \infty$. In other words, $(G_n)_{n\in\mathbb{N}}$ is FO-convergent if the probability that a formula ϕ is satisfied by the graph G_n with a random assignment of vertices of G_n to the free variables

of ϕ converges as n grows to infinity. We also consider analogously defined X-convergence, where X is a fragment of FO.

Our main result is that this model of FO-convergence is a suitable model for the analysis of limits of graphs with bounded tree depth. This class of graphs can be defined by logical terms as well as combinatorially in various ways. The most concise definition is that a class of graphs has bounded tree depth if and only if the maximal length of a path in every G in the class is bounded by a constant. Tree depth is related to the quantifier depth of a formula and to other combinatorial characteristics, see [50] and [61] for a full discussion. The convergence of graphs with bounded tree depth is analysed in detail and this leads to explicit limits for those sequences of graphs where all members of the sequence have uniformly bounded tree depth (see Theorem 27). One may argue that this is a rather special class of graphs. But we believe that this is not so and that there is here more than meets the eye. Let us outline the reasons for this optimism: For graphs, and more generally for finite structures, there is a class dichotomy: *nowhere dense* and *somewhere dense* [55, 57]. Each class of graphs falls in one of these two categories. Somewhere dense class $\mathcal{C}$ may be characterised by saying that there exists a (primitive positive) FO interpretation of all graphs into them. Such class $\mathcal{C}$ is inherently a class of dense graphs. On the other hand any nowhere dense class of structures may by covered by a few subgraphs with small tree depth. This is called *low tree-depth decomposition* and it found many applications in structural graphs theory, see e.g. [61] for more information about this development. Thus graphs with bounded tree depth form building blocks of graphs in a nowhere dense class. So in this setting the solution of limits for graphs with bounded tree depth presents a step in solving the general limits for sparse graphs.

To create a proper model for bounded height trees we have to introduce the model in a greater generality and it appeared that our approach relates and in most cases generalizes all existing models of graph limits.

Our unified framework to study structural limits of general relational structures and limits of graphs (in particular) via X-convergence generalizes most instances of graph limits considered previously. For instance, for the fragment X of all existential first-order formulas, X-convergence means that the probability that a structure has any given extension property converges. Our approach is encouraged by the deep connections to four notions of convergence which have been proposed to study graph limits in different contexts.

The first of these is the combinatorially motivated notion of convergence introduced by Lovász and Szegedy [46] and further developed by Borgs, Chayes, Lovász, Sós and Vesztergombi [11, 12]. This notion is established by equipping the space of (unlabelled) graphs with a suitable metric, and by considering convergent sequences defined as Cauchy sequences in the completion of this metric space. In this setting, a classic representation of the limit [46, 11] is a symmetric Lebesgue measurable function $W : [0,1]^2 \to [0,1]$ called *graphon*. Such a representation is of course not unique, in the sense that different graphons may define the same graph limit [8, 16]. A connection between graph limits and de Finetti's theorem for exchangeable arrays (and the early works of Aldous [2], Hoover [37] and Kallenberg [38]) has been established, see e.g. Diaconis and Janson [16]. This convergence corresponds to X-convergence with respect to the fragment of all quantifier free formulas (see Section 4.1).

A second approach to graph limits, which concerns specifically connected graphs with bounded degrees, has been developed by Benjamini and Schramm [6] and Aldous [3]. In this case, the limit of a convergent sequence is described as an unimodular distribution of the space of rooted connected countable graphs with

bounded degree (equipped with a suitable metric and the derived σ-algebra). A representation of the limit is a *measurable graphing* (notion introduced by Adams [1] in the context of Ergodic theory), that is a standard Borel space with a measure μ and d measure preserving Borel involutions. The existence of such a representation has been made explicit by Elek [23], and relies on the works of Benjamini [6] and Gaboriau [29]. This convergence corresponds to X-convergence with respect to the fragment of all local formulas (see Section 4.2).

A third (but earlier) theory of limits of structures, by means of *elementary convergence*, derives from two important results in first-order logic, namely Gödel's completeness theorem and the compactness theorem. This notion of convergence, based on the satisfaction of first-order sentences, can also be expressed by embedding graphs in the space of complete theories (in the language of graphs) and by equipping this space with an ultrametric derived from the quantifier rank. In this setting, a limit is a complete theory, which can be represented by a model and, in this case, the limit of a sequence of graphs can be represented by a countable graph. This convergence corresponds to X-convergence with respect to the fragment of all sentences (see Section 4.3).

Finally, a notion of limit developed by the authors has been introduced, which is based on the existence of homomorphisms from small graphs [58, 61].

In this paper, we shed a new light on model theoretical constructions by an approach inspired by functional analysis. The preliminary material and our framework are introduced in Sections 2 and 3. The general approach presented in the first sections of this paper leads to several new results. Some of them intuitively motivate and, we believe, justify the introduction of first-order formulas. Let us mention a sample of such results.

Graph limits (in the sense of Lovász et al.) — and more generally hypergraph limits — have been studied by Elek and Szegedy [24] through the introduction of a measure on the ultraproduct of the graphs in the sequence (via Loeb measure construction, see [42]). The fundamental theorem of ultraproducts proved by Łoś [43] implies that the ultralimit of a sequence of graphs is (as a measurable graph) an FO-limit. Thus in this non-standard setting we get FO-limits (almost) for free.

Central to the theory of graph limits stand random graphs (in the Erdős-Rényi model [25]): a sequence of random graphs with increasing order and edge probability $0 < p < 1$ is almost surely convergent to the constant graphon p [46]. On the other hand, it follows from the work of Erdős and Rényi [26] that such a sequence is almost surely elementarily convergent to an ultra-homogeneous graph, called the *Rado graph*. We prove that these two facts, together with the quantifier elimination property of ultra-homogeneous graphs, imply that a sequence of random graphs with increasing order and edge probability $0 < p < 1$ is almost surely FO-convergent, see Section 5.3. (However, it is presently open whether a representation of the limit exists, that would be as nice as a graphon.)

We shall also prove that a sequence of bounded degree graphs $(G_n)_{n\in\mathbb{N}}$ with $|G_n| \to \infty$ is FO-convergent if and only if it is both convergent in the sense of Benjamini-Schramm and in the sense of elementary convergence, and that the limit can still be represented by a graphing, see Sections 4.2 and 7.4.

We prove that the limit of an FO-convergent sequence of graphs is a probability measure on the Stone space of the Boolean algebra of first-order formulas, which is invariant under the action of S_ω on this space, see Section 3. Fine interplay of these notions is depicted on Table 1.

One of the main issue of our general approach is to determine a representation of FO-limits as measurable graphs. A natural limit object is a standard probability space (V, Σ, μ) together with a graph with vertex set V and edge set E, with the

Boolean algebra $\mathcal{B}(X)$	Stone Space $S(\mathcal{B}(X))$
Formula ϕ	Continuous function f_ϕ
Vertex v	"Type of vertex" T
Graph G	statistics of types =probability measure μ_G
$\langle \phi, G\rangle$	$\int f_\phi(T)\ \mathrm{d}\mu_G(T)$
X-convergent (G_n)	weakly convergent μ_{G_n}
$\Gamma = \mathrm{Aut}(\mathcal{B}(X))$	Γ-invariant measure

TABLE 1. Some correspondances

property that every first-order definable subset of a power of V is measurable. This leads to the notions of relational sample space and modeling. Relying on our general approach, we consider the problem of the representation of FO-convergent sequences of graphs with bounded diameter connected components, and specifically the representation of FO-convergent sequences of graphs with bounded tree-depth [50]. This study is motivated by the structural simplicity of these graphs and the central role they play in the context of sparse graph decompositions [51, 52, 53, 61] and in first-order quantifier elimination schemes for bounded expansion classes of graphs [20, 32]. The special case of FO-limits of graphs with bounded tree-depth should be regarded as a first step toward a representation of FO-limits of graphs belonging to arbitrary fixed bounded expansion class, like planar graphs, or graphs excluding a topological minor. Combining low tree-depth decomposition techniques with the results of this paper gives us a "road map" to attack the general classes of sparse graphs (see remarks in the last section of this paper).

The class of graphs with bounded tree-depth is one of the first "intermediate cases" [44, 45] for which one can explicitly describe a limit object. The only other case we are aware is Lyons [47], which is concerned by a slight relaxation of the condition to have bounded degree to the condition that — roughly speaking — large degrees are far away from almost all other vertices (so that the asymptotic distribution of rooted finite graphs found at a random vertex converges to a probability distribution); this does not apply, for instance, for sequences of connected graphs with bounded diameters. Also, the scaling limit of vertex transitive graphs with sufficiently fast growing diameter has been characterized recently [5], but scaling limits and structural limits are in some sense orthogonal concepts.

We believe that the approach taken in this paper is natural and that it enriches the existing notions of limits. It also presents, for example via decomposition techniques (low-tree depth decomposition, see [61]) a promising approach to more general intermediate classes (see the final comments).

The outline of the paper is as follows:

- In Section 2 we introduce main definitions and results of the paper.
- In Section 3 we recall some relevant properties of Boolean algebras. To each Boolean algebra B — considered as a algebra of sets — we associate

the Banach space $\mathrm{ba}(B)$ of all bounded and finitely additive measures on B. We note that $\mathrm{ba}(B)$ is isometrically isomorphic to the continuous dual $\mathrm{rca}(S(B))$ of the Banach $C(S(B))$ of the continuous real-valued functions defined on the Stone space $S(B)$ dual to B, and deduce that the space additive functions on B (with pointwise convergence) is isomorphic to the space of regular countably additive measures on $S(B)$ (with weak-$*$ convergence). Then we show that if a notion of independence is introduced in B together with a suitable group action, then we can deduce that pointwise limits of symmetric normalized positive additive functions on B that are multiplicative on independent elements correspond to symmetric probabilities on the Stone space of B allowing particular disintegrations. Then we recall some basics of model theory and introduce our unified framework for structural limits. We introduce the pairing of a first-order formula and a finite structure and the notion of X-convergence for a fragment X of FO and we translate our results to this framework. Thus this correspondence is an alternative and very general approach to measure theoretic aspects of limits.

- In Section 4 we express the classical notions of limits in terms of convergences controlled by proper fragments of FO and deduce the particular role of elementary convergence. We further study the properties of the Stone spaces associated to diverse fragments of FO. Under the assumption that a sequence of connected graphs with degree at most D (with order tending to infinity) is elementary convergent, we prove that FO-convergence is equivalent to Benjamini-Schramm convergence. Also, we prove that under the assumption that a sequence of graphs is elementary convergent to a countable ultra-homogeneous graph the FO-convergence is equivalent to Lovász convergence. Note that this condition on the elementary limit is not as strong as it may seem at first glance, as random graphs almost surely converge elementarily to the Rado graph.
- In Section 5 we study how different fragments combine. For instance, we study the FO^p-hierarchy, and the particular roles of local formulas and sentences.
- In Section 6 we consider a non-standard approach to construct a limit object.
- In Section 7 we introduce the notions of relational sample space and of modeling. We prove that in the particular case of graphs with bounded degrees, the notion of modeling coincides with the notion of graphing, and that every FO-convergent sequence admits a modeling as a limit.
- In Section 8 we study how modeling limits of FO-convergent sequences can be merged into a modeling limit of the index-wise union of the sequences.
- In Section 9 we study limits of sequences of disconnected graphs and address the problem of tracking the connected components till the limit.
- In Section 10 we focus on the the class of rooted colored trees with bounded height. We give an explicit description of a relational sample space of a modeling FO-limit for graphs in this class.
- In Section 11 we introduce the class of graphs with bounded tree-depth and show how a proper encoding and the use of a basic interpretation scheme can be used to reduce the problem of finding a modeling FO-limit for FO-convergent sequences of graphs with uniformly bounded tree-depth to a similar problem on vertex-colored rooted trees with uniformly bounded height.
- Section 12 is devoted to concluding remarks and discussion.

2. Main Definitions and Results

If we consider relational structures with signature λ, the symbols of the relations and constants in λ define the non-logical symbols of the vocabulary of the first-order language $\mathrm{FO}(\lambda)$ associated to λ-structures. Notice that if λ is at most countable then $\mathrm{FO}(\lambda)$ is countable. The symbols of variables will be assumed to be taken from a countable set $\{x_1, \dots, x_n, \dots\}$ indexed by $\mathbb{N}$. Let $u_1, \dots, u_k$ be terms. The set of used free variables of a formula ϕ will be denoted by $\mathrm{Fv}(\phi)$ (by saying that a variable x_i is "used" in ϕ we mean that ϕ is not logically equivalent to a formula in which x_i does not appear). The formula $\phi_{x_{i_1},\dots,x_{i_k}}(u_1, \dots, u_k)$ denote the formula obtained by substituting simultaneously the term u_j to the free occurences of x_{i_j} for $j = 1, \dots, k$. In the sake of simplicity, we will denote by $\phi(u_1, \dots, u_k)$ the substitution $\phi_{x_1,\dots,x_k}(u_1, \dots, u_k)$.

A *relational structure* $\mathbf{A}$ with signature λ is defined by its *domain* (or *universe*) A and relations with names and arities as defined in λ. In the following we will denote relational structures by bold face letters $\mathbf{A}, \mathbf{B}, \dots$ and their domains by the corresponding light face letters $A, B, \dots$

The key to our approach are the following two definitions.

Definition 1 (Stone pairing)**.** Let λ be a signature, let $\phi \in \mathrm{FO}(\lambda)$ be a first-order formula with free variables $x_1, \dots, x_p$ and let $\mathbf{A}$ be a finite λ-structure.

Let
$$\Omega_\phi(\mathbf{A}) = \{(v_1, \dots, v_p) \in A^p : \ \mathbf{A} \models \phi(v_1, \dots, v_p)\}.$$

We define the *Stone pairing* of ϕ and $\mathbf{A}$ by

$$\langle \phi, \mathbf{A} \rangle = \frac{|\Omega_\phi(\mathbf{A})|}{|A|^p}. \tag{1}$$

In other words, $\langle \phi, \mathbf{A} \rangle$ is the probability that ϕ is satisfied in $\mathbf{A}$ when we interpret the p free variables of ϕ by p vertices of G chosen randomly, uniformly and independently.

Note that in the case of a sentence ϕ (that is a formula with no free variables, thus $p = 0$), the definition of the Stone pairing reduces to

$$\langle \phi, \mathbf{A} \rangle = \begin{cases} 1, & \text{if } \mathbf{A} \models \phi; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2 (FO-convergence)**.** A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of finite λ-structures is *FO-convergent* if, for every formula $\phi \in \mathrm{FO}(\lambda)$ the sequence $(\langle \phi, \mathbf{A}_n \rangle)_{n\in\mathbb{N}}$ is (Cauchy) convergent.

In other words, a sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is FO-convergent if the sequence of mappings $\langle \cdot, \mathbf{A}_n \rangle : \mathrm{FO}(\lambda) \to [0, 1]$ is pointwise-convergent.

The interpretation of the Stone pairing as a probability suggests to consider finite λ-structures as particular probability spaces and to extend this view to more general λ-structures.

Definition 3 (Relational sample space)**.** A *relational sample space* is a relational structure $\mathbf{A}$ (with signature λ) with extra structure: The domain A of $\mathbf{A}$ of a sample model is a standard Borel space (with Borel σ-algebra Σ_A) with the property that every subset of A^p that is first-order definable in $\mathrm{FO}(\lambda)$ is measurable (in A^p with respect to the product σ-algebra).

In other words, if $\mathbf{A}$ is a relational sample space then for every integer p and every $\phi \in \mathrm{FO}(\lambda)$ with p free variables it holds $\Omega_\phi(\mathbf{A}) \in \Sigma_{\mathbf{A}}^p$.

Definition 4 (Modeling)**.** A *modeling* $\mathbf{A}$ is a relational sample space equipped with a probability measure (denoted $\mu_{\mathbf{A}}$).

A *modeling* with signature λ is a λ*-modeling.*

Remark 1. We take time for some comments on the above definitions:

- According to Kuratowski's isomorphism theorem, the domains of relational sample spaces are isomorphic to either $\mathbb{R}$, $\mathbb{Z}$, or a finite space.
- *Borel graphs* (in the sense of Kechris et al. [39]) are generally not modelings (in our sense) as Borel graphs are only required to have a measurable adjacency relation.
- By equipping its domain with the discrete σ-algebra, every finite λ-structure defines with a relational sample space. Considering the uniform probability measure on this space then canonically defines a weakly uniform modeling.
- It follows immediately from Definition 3 that any *k-rooting* of a relational sample space is a relational sample space.

We extend the definition of Stone pairing to modelings as follows.

Definition 5 (Stone pairing)**.** Let λ be a signature, let $\phi \in \mathrm{FO}(\lambda)$ be a first-order formula with free variables $x_1, \dots, x_p$ and let $\mathbf{A}$ be a λ-modeling.

We define the *Stone pairing* of ϕ and $\mathbf{A}$ by

$$\langle \phi, \mathbf{A} \rangle = \int_{x \in A^p} 1_{\Omega_\phi(\mathbf{A})}(x)\, \mathrm{d}\nu_{\mathbf{A}}^p(x). \tag{2}$$

Note that the definition of a modeling is simply forged to make the expression (2) meaningful. Based on this definition, modelings can sometimes be used as a representation of the limit of an FO-convergent sequence of finite λ-structures.

Definition 6. A modeling $\mathbf{L}$ is a *modeling* FO*-limit* of an FO-convergent sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of finite λ-structures if $\langle\, \cdot\,, \mathbf{A}_n\rangle$ converges pointwise to $\langle\, \cdot\,, \mathbf{L}\rangle$.

As we shall see in Lemma 17, a modeling FO-limit of an FO-convergent sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of finite λ-structures is necessarily weakly uniform. It follows that if a modeling $\mathbf{L}$ is a modeling FO-limit then L is either finite or uncountable.

It is not clear whether every FO-convergent sequence of finite relational structures admits a modeling FO-limit, and we strongly believe this is not the case. However, we prove that modeling FO-limits exist in two particular cases.

Theorem 1. *Let C be a integer.*

(1) Every FO-convergent sequence of graphs with maximum degree at most C has a modeling FO-limit;

(2) Every FO-convergent sequence of rooted trees with height at most C has a modeling FO-limit.

While the first item will be derived from the graphing representation of limits of Benjamini-Schramm convergent sequences of graphs with bounded maximum degree with no major difficulties, the second item will be quite difficult to establish and is the main result of this paper.

As mentioned earlier, we do not know if a modeling FO-limit exists in general. However, a non-standard approach allows to define a limit object by relaxing the requirement that the domain of the limit structure is a standard probability space and by replacing — in the definition of the Stone pairing — the integration the product space by an iterated integration on the domain.

Theorem 2. *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be an FO-convergent sequence of λ-structures. Then there exists a λ-structure $\mathbf{L}$ such that the domain of L is a (non-separable) probability space with probability measure ν, such that for every first-order formula $\phi \in \mathrm{FO}(\lambda)$ with free variables $x_1, \dots, x_p$ it holds:*

$$\lim_{n\to\infty} \langle \phi, \mathbf{A}_n \rangle = \int \cdots \int_{A} 1_{\Omega_\phi(\mathbf{A})}(x_1, \dots, x_p)\ \mathrm{d}\nu(x_1)\ \dots\ \mathrm{d}\nu(x_p).$$

This kind of limit object is however not completely satisfactory, and we shall usually prefer to have a general representation of the limit of an FO-convergent sequence of λ-structures by means of a probability distribution on a compact Polish space S_λ defined from $\mathrm{FO}(\lambda)$ using Stone duality. In this context we have the following result:

Theorem 3. *Let λ be a fixed finite or countable signature. Then there exist two mappings $\mathbf{A} \mapsto \mu_{\mathbf{A}}$ and $\phi \mapsto K(\phi)$ such that*

- *$\mathbf{A} \mapsto \mu_{\mathbf{A}}$ is an injective mapping from the class of finite λ-structures to the space of regular probability measures on S_λ,*
- *$\phi \mapsto K(\phi)$ is a mapping from $\mathrm{FO}(\lambda)$ to the set of the clopen subsets of S_λ,*

such that for every finite λ-structure $\mathbf{A}$ and every first-order formula $\phi \in \mathrm{FO}(\lambda)$ it holds:

$$\langle \phi, \mathbf{A} \rangle = \int_{S_\lambda} 1_{K(\phi)}\, \mathrm{d}\mu_{\mathbf{A}}.$$

(To prevent risks of notational ambiguity, we shall use μ as root symbol for measures on Stone spaces and keep ν for measures on modelings.)

Consider an FO-convergent sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$. Then the pointwise convergence of $\langle \cdot, \mathbf{A}_n \rangle$ translates as a weak $*$-convergence of the measures $\mu_{\mathbf{A}_n}$ and we get:

Theorem 4. *If $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is an FO-convergent sequence of finite λ-structures there exists a unique regular probability measure μ on S_λ such that for every first-order formula $\phi \in \mathrm{FO}(\lambda)$ it holds:*

$$\lim_{n\to\infty} \langle \phi, \mathbf{A}_n \rangle = \int_{S_\lambda} 1_{K(\phi)}\, \mathrm{d}\mu.$$

3. Limits as Measures on Stone Spaces

In order to prove the representation theorems Theorem 3 and Theorem 4, we first need to prove a general representation for additive functions on Boolean algebras.

3.1. **Representation of Additive Functions.** Recall that a *Boolean algebra* $B = (B, \wedge, \vee, \neg, 0, 1)$ is an algebra with two binary operations $\vee$ and $\wedge$, a unary operation $\neg$ and two elements 0 and 1, such that $(B, \vee, \wedge)$ is a complemented distributive lattice with minimum 0 and maximum 1. The two-elements Boolean algebra is denoted $\mathbf{2}$.

To a Boolean algebra B is associated a topological space, denoted $S(B)$, whose points are the ultrafilters on B (or equivalently the homomorphisms $B \to \mathbf{2}$). The topology on $S(B)$ is generated by a sub-basis consisting of all sets

$$K_B(b) = \{x \in S(B) : b \in x\},$$

where $b \in B$. When the considered Boolean algebra will be clear from context we shall omit the subscript and write $K(b)$ instead of $K_B(b)$.

A topological space is a *Stone space* if it is Hausdorff, compact, and has a basis of clopen subsets. Boolean spaces and Stone spaces are equivalent as formalized by Stone representation theorem [64], which states (in the language of category theory) that there is a duality between the category of Boolean algebras (with homomorphisms) and the category of Stone spaces (with continuous functions). This justifies to call $S(B)$ the *Stone space* of the Boolean algebra B. The two contravariant functors defining this duality are denoted by S and Ω and defined as follows:

For every homomorphism $h : A \to B$ between two Boolean algebra, we define the map $S(h) : S(B) \to S(A)$ by $S(h)(g) = g \circ h$ (where points of $S(B)$ are identified with homomorphisms $g : B \to \mathbf{2}$). Then for every homomorphism $h : A \to B$, the map $S(h) : S(B) \to S(A)$ is a continuous function.

Conversely, for every continuous function $f : X \to Y$ between two Stone spaces, define the map $\Omega(f) : \Omega(Y) \to \Omega(X)$ by $\Omega(f)(U) = f^{-1}(U)$ (where elements of $\Omega(X)$ are identified with clopen sets of X). Then for every continuous function $f : X \to Y$, the map $\Omega(f) : \Omega(Y) \to \Omega(X)$ is a homomorphism of Boolean algebras.

We denote by $K = \Omega \circ S$ one of the two natural isomorphisms defined by the duality. Hence, for a Boolean algebra B, $K(B)$ is the set algebra $\{K_B(b) : b \in B\}$, and this algebra is isomorphic to B.

An ultrafilter of a Boolean algebra B can be considered as a finitely additive measure, for which every subset has either measure 0 or 1. Because of the equivalence of the notions of Boolean algebra and of set algebra, we define the space $\mathrm{ba}(B)$ as the space of all bounded additive functions $f : B \to \mathbb{R}$. Recall that a function $f : B \to \mathbb{R}$ is *additive* if for all $x, y \in B$ it holds

$$x \wedge y = 0 \quad \Longrightarrow \quad f(x \vee y) = f(x) + f(y).$$

The space $\mathrm{ba}(B)$ is a Banach space for the norm

$$\|f\|_{\mathrm{ba}(B)} = \sup_{x \in B} f(x) - \inf_{x \in B} f(x).$$

(Recall that the ba space of an algebra of sets Σ is the Banach space consisting of all bounded and finitely additive measures on Σ with the total variation norm.)

Let $V(B)$ be the normed vector space (of so-called *simple functions*) generated by the indicator functions of the clopen sets (equipped with supremum norm). The indicator function of clopen set $K(b)$ (for some $b \in B$) is denoted by $\mathbf{1}_{K(b)}$.

Lemma 1. *The space* $\mathrm{ba}(B)$ *is the topological dual of* $V(B)$

Proof. One can identify $\mathrm{ba}(B)$ with the space $\mathrm{ba}(K(B))$ of finitely additive measure defined on the set algebra $K(B)$. As a vector space, $\mathrm{ba}(B) \approx \mathrm{ba}(K(B))$ is then clearly the (algebraic) dual of the normed vector space $V(B)$.

The pairing of a function $f \in \mathrm{ba}(B)$ and a vector $X = \sum_{i=1}^n a_i \mathbf{1}_{K(b_i)}$ is defined by

$$[f, X] = \sum_{i=1}^n a_i f(b_i).$$

That $[f, X]$ does not depend on a particular choice of a decomposition of X follows from the additivity of f. We include a short proof for completeness: Assume

$\sum_i \alpha_i \mathbf{1}_{K(b_i)} = \sum_i \beta_i \mathbf{1}_{K(b_i)}$. As for every $b, b' \in B$ it holds $f(b) = f(b \wedge b') + f(b \wedge \neg b')$ and $\mathbf{1}_{K(b)} = \mathbf{1}_{K(b\wedge b')} + \mathbf{1}_{K(b \wedge \neg b')}$ we can express the two sums as $\sum_j \alpha'_j \mathbf{1}_{K(b'_j)} = \sum_j \beta'_j \mathbf{1}_{K(b'_j)}$ (where $b'_i \wedge b'_j = 0$ for every $i \neq j$), with $\sum_i \alpha_i f(b_i) = \sum_j \alpha'_j f(b'_j)$ and $\sum_i \beta_i f(b_i) = \sum_j \beta'_j f(b'_j)$. As $b'_i \wedge b'_j = 0$ for every $i \neq j$, for $x \in K(b'_j)$ it holds $\alpha'_j = X(x) = \beta'_j$. Hence $\alpha'_j = \beta'_j$ for every j. Thus $\sum_i \alpha_i f(b_i) = \sum_i \beta_i f(b_i)$.

Note that $X \mapsto [f, X]$ is indeed continuous. Thus $\mathrm{ba}(B)$ is the topological dual of $V(B)$. □ □

Lemma 2. *The vector space $V(B)$ is dense in $C(S(B))$ (with the uniform norm).*

Proof. Let $f \in C(S(B))$ and let $\epsilon > 0$. For $z \in f(S(B))$ let U_z be the preimage by f of the open ball $B_{\epsilon/2}(z)$ of $\mathbb{R}$ centered in z. As f is continuous, U_z is a open set of $S(B)$. As $\{K(b) : b \in B\}$ is a basis of the topology of $S(B)$, U_z can be expressed as a union $\bigcup_{b \in \mathcal{F}(U_z)} K(b)$. It follows that $\bigcup_{z \in f(S(B))} \bigcup_{b \in \mathcal{F}(U_z)} K(b)$ is a covering of $S(B)$ by open sets. As $S(B)$ is compact, there exists a finite subset $\mathcal{F}$ of $\bigcup_{z\in f(S(B))} \mathcal{F}(U_z)$ that covers $S(B)$. Moreover, as for every $b, b' \in B$ it holds $K(b) \cap K(b') = K(b \wedge b')$ and $K(b) \setminus K(b') = K(b \wedge \neg b')$ it follows that we can assume that there exists a finite family $\mathcal{F}'$ such that $S(B)$ is covered by open sets $K(b)$ (for $b \in \mathcal{F}'$) and such that for every $b \in \mathcal{F}'$ there exists $b' \in \mathcal{F}$ such that $K(b) \subseteq K(b')$. In particular, it follows that for every $b \in \mathcal{F}'$, $f(K(b))$ is included in an open ball of radius $\epsilon/2$ of $\mathbb{R}$. For each $b \in \mathcal{F}'$ choose a point $x_b \in S(B)$ such that $b \in x_b$. Now define

$$\hat{f} = \sum_{b \in \mathcal{F}'} f(x_b) \mathbf{1}_{K(b)}$$

Let $x \in S(B)$. Then there exists $b \in \mathcal{F}'$ such that $x \in K(b)$. Thus

$$|f(x) - \hat{f}(x)| = |f(x) - f(x_b)| < \epsilon.$$

Hence $\|f - \hat{f}\|_\infty < \epsilon$. □ □

Lemma 3. *Let B be a Boolean algebra, let $\mathrm{ba}(B)$ be the Banach space of bounded additive real-valued functions equipped with the norm $\|f\| = \sup_{b\in B} f(b) - \inf_{b \in B} f(b)$, let $S(B)$ be the Stone space associated to B by Stone representation theorem, and let $\mathrm{rca}(S(B))$ be the Banach space of the regular countably additive measure on $S(B)$ equipped with the total variation norm.*

Then the mapping $C_K : \mathrm{rca}(S(B)) \to \mathrm{ba}(B)$ defined by $C_K(\mu) = \mu \circ K$ is an isometric isomorphism. In other words, C_K is defined by

$$C_K(\mu)(b) = \mu(\{x \in S(B) : \ b \in x\})$$

(considering that the points of $S(B)$ are the ultrafilters on B).

Proof. According to Lemma 1, the Banach space $\mathrm{ba}(B)$ is the topological dual of $V(B)$ and as $V(B)$ is dense in $C(S(B))$ (according to Lemma 2) we deduce that $\mathrm{ba}(B)$ can be identified with the continuous dual of $C(S(B))$. By Riesz representation theorem, the topological dual of $C(S(B))$ is the space $\mathrm{rca}(S(B))$ of regular countably additive measures on $S(B)$. From these observations follows the equivalence of $\mathrm{ba}(B)$ and $\mathrm{rca}(S(B))$.

This equivalence is easily made explicit, leading to the conclusion that the mapping $C_K : \mathrm{rca}(S(B)) \to \mathrm{ba}(B)$ defined by $C_K(\mu) = \mu \circ K$ is an isometric isomorphism. □ □

Note also that, similarly, the restriction of C_K to the space $\Pr(S(B))$ of all (regular) probability measures on $S(B)$ is an isometric isomorphism of $\Pr(S(B))$ and the subset $\mathrm{ba}_1(B)$ of $\mathrm{ba}(B)$ of all positive additive functions f on B such that $f(1) = 1$.

Recall that given a measurable function $f : X \to Y$ (where X and Y are measurable spaces), the *pushforward* $f_*(\mu)$ of a measure μ on X is the measure on Y defined by $f_*(\mu)(A) = \mu(f^{-1}(A))$ (for every measurable set A of Y). Note that if f is a continuous function and if μ is a regular measure on X, then the pushforward measure $f_*(\mu)$ is a regular measure on Y. By similarity with the definition of $\Omega(f) : \Omega(Y) \to \Omega(X)$ (see above definition) we denote by $\Omega_*(f)$ the mapping from $\mathrm{rca}(X)$ to $\mathrm{rca}(Y)$ defined by $(\Omega_*(f))(\mu) = f_*(\mu)$.

All the functors defined above are consistent in the sense that if $h : A \to B$ is a homomorphism and $f \in \mathrm{ba}(B)$ then

$$\Omega_*(S(h))(\mu_f) \circ K_A = f \circ h = \tau_h(f).$$

A standard notion of convergence in $\mathrm{rca}(S(B))$ (as the continuous dual of $C(S(B))$) is the weak $*$-convergence: a sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures is convergent if, for every $f \in C(S(B))$ the sequence $\int f(x)\,\mathrm{d}\mu_n(x)$ is convergent. Thanks to the density of $V(B)$ this convergence translates as pointwise convergence in $\mathrm{ba}(B)$ as follows: a sequence $(g_n)_{n\in\mathbb{N}}$ of functions in $\mathrm{ba}(B)$ is convergent if, for every $b \in B$ the sequence $(g_n(b))_{n\in\mathbb{N}}$ is convergent. As $\mathrm{rca}(S(B))$ is complete, so is $\mathrm{rca}(B)$. Moreover, it is easily checked that $\mathrm{ba}_1(B)$ is closed in $\mathrm{ba}(B)$.

In a more concise way, we can write, for a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathrm{ba}(B)$ and for the corresponding sequence $(\mu_{f_n})_{n\in\mathbb{N}}$ of regular measures on $S(B)$:

$$\lim_{n\to\infty} f_n \text{ pointwise} \qquad \iff \qquad \mu_{f_n} \Rightarrow \mu_f.$$

3.2. **Basics of Model Theory and Lindenbaum-Tarski Algebras.** We denote by $\mathcal{B}(\mathrm{FO}(\lambda))$ the equivalence classes of $\mathrm{FO}(\lambda)$ defined by logical equivalence. The (class of) unsatisfiable formulas (resp. of tautologies) will be designated by 0 (resp. 1). Then, $\mathcal{B}(\mathrm{FO}(\lambda))$ gets a natural structure of Boolean algebra (with minimum 0, maximum 1, infimum $\wedge$, supremum $\vee$, and complement $\neg$). This algbera is called the *Lindenbaum-Tarski algebra* of $\mathrm{FO}(\lambda)$. Notice that all the Boolean algebras $\mathrm{FO}(\lambda)$ for countable λ are isomorphic, as there exists only one countable atomless Boolean algebra up to isomorphism (apparently proved by Tarski; see also [35]).

For an integer $p \geq 1$, the fragment $\mathrm{FO}_p(\lambda)$ of $\mathrm{FO}(\lambda)$ contains first-order formulas ϕ such that $\mathrm{Fv}(\phi) \subseteq \{x_1, \dots, x_p\}$. The fragment $\mathrm{FO}_0(\lambda)$ of $\mathrm{FO}(\lambda)$ contains first-order formulas without free variables (that is *sentences*).

We check that the permutation group S_p on $[p]$ acts on $\mathrm{FO}_p(\lambda)$ by $\sigma \cdot \phi = \phi(x_{\sigma(1)}, \dots, x_{\sigma(p)})$ and that each permutation indeed define an automorphism of $\mathcal{B}(\mathrm{FO}_p(\lambda))$. Similarly, the group S_ω of permutation on $\mathbb{N}$ with finite support acts on $\mathrm{FO}(\lambda)$ and $\mathcal{B}(\mathrm{FO}(\lambda))$. Note that $\mathrm{FO}_0(\lambda) \subseteq \dots \subseteq \mathrm{FO}_p(\lambda) \subseteq \mathrm{FO}_{p+1}(\lambda) \subseteq \dots \subseteq \mathrm{FO}(\lambda)$. Conversely, let $\mathrm{rank}(\phi) = \max\{i : x_i \in \mathrm{Fv}(\phi)\}$. Then we have a natural projection $\pi_p : \mathrm{FO}(\lambda) \to \mathrm{FO}_p(\lambda)$ defined by

$$\pi_p(\phi) = \begin{cases} \phi & \text{if } \mathrm{rank}(\phi) \leq p \\ \exists x_{p+1}\, \exists x_{p+2}\, \dots\, \exists x_{\mathrm{rank}(\phi)}\, \phi & \text{otherwise} \end{cases}$$

An *elementary class* (or *axiomatizable class*) $\mathcal{C}$ of λ-structures is a class consisting of all λ-structures satisfying a fixed consistent first-order theory $T_\mathcal{C}$. Denoting by $\mathcal{I}_{T_\mathcal{C}}$ the ideal of all first-order formulas in $\mathcal{L}$ that are provably false from axioms in $T_\mathcal{C}$, The Lindenbaum-Tarski algebra $\mathcal{B}(\mathrm{FO}(\lambda), T_\mathcal{C})$ associated to the theory $T_\mathcal{C}$ of $\mathcal{C}$ is the quotient Boolean algebra $\mathcal{B}(\mathrm{FO}(\lambda), T_\mathcal{C}) = \mathcal{B}(\mathrm{FO}(\lambda))/\mathcal{I}_{T_\mathcal{C}}$. As a set, $\mathcal{B}(\mathrm{FO}(\lambda), T_\mathcal{C})$ is simply the quotient of $\mathrm{FO}(\lambda)$ by logical equivalence modulo $T_\mathcal{C}$.

As we consider countable languages, $T_\mathcal{C}$ is at most countable and it is easily checked that $S(\mathcal{B}(\mathrm{FO}(\lambda), T_\mathcal{C}))$ is homeomorphic to the compact subspace of $S(\mathcal{B}(\mathrm{FO}(\lambda))$ defined as $\{T \in S(\mathcal{B}(\mathrm{FO}(\lambda))) : T \supseteq T_\mathcal{C}\}$. Note that, for instance,

$S(\mathcal{B}(\mathrm{FO}_0(\lambda), T_{\mathcal{C}}))$ is a clopen set of $S(\mathcal{B}(\mathrm{FO}_0(\lambda)))$ if and only if $\mathcal{C}$ is *finitely axiomatizable* (or a *basic* elementary class), that is if $T_{\mathcal{C}}$ can be chosen to be a single sentence. These explicit correspondences are particularly useful to our setting.

3.3. **Stone Pairing Again.** We take some time to comment Definition 5. Note first that this definition is consistent in the sense that for every modeling $\mathbf{A}$ and for every formula $\phi \in \mathrm{FO}(\lambda)$ with p free variables can be considered as a formula with $q \geq p$ free variables with $q-p$ unused variables, we have

$$\int_{A^q} 1_{\Omega_\phi(\mathbf{A})}(x)\,\mathrm{d}\nu_{\mathbf{A}}^q(x) = \int_{A^p} 1_{\Omega_\phi(\mathbf{A})}(x)\,\mathrm{d}\nu_{\mathbf{A}}^p(x).$$

It is immediate that for every formula ϕ it holds $\langle\neg\phi, \mathbf{A}\rangle = 1 - \langle\phi, \mathbf{A}\rangle$. Moreover, if $\phi_1, \dots, \phi_n$ are formulas, then by de Moivre's formula, it holds

$$\langle\bigvee_{i=1}^n \phi_i, \mathbf{A}\rangle = \sum_{k=1}^n (-1)^{k+1}\Bigg(\sum_{1\leq i_1<\dots<i_k\leq n} \langle\bigwedge_{j=1}^k \phi_{i_j}, \mathbf{A}\rangle\Bigg).$$

In particular, if $\phi_1, \dots, \phi_k$ are *mutually exclusive* (meaning that $\phi_i \wedge \phi_j = 0$) then it holds

$$\langle\bigvee_{i=1}^k \phi_i, \mathbf{A}\rangle = \sum_{i=1}^k \langle\phi_i, \mathbf{A}\rangle.$$

It follows that for every fixed modeling $\mathbf{A}$, the mapping $\phi \mapsto \langle\phi, \mathbf{A}\rangle$ is additive (i.e. $\langle\,\cdot\,, \mathbf{A}\rangle \in \mathrm{ba}(\mathcal{B}(\mathrm{FO}(\lambda)))$):

$$\phi_1 \wedge \phi_2 = 0 \quad\Longrightarrow\quad \langle\phi_1 \vee \phi_2, \mathbf{A}\rangle = \langle\phi_1, \mathbf{A}\rangle + \langle\phi_2, \mathbf{A}\rangle.$$

The Stone pairing is antimonotone:

Let $\phi, \psi \in \mathrm{FO}(\lambda)$. For every modeling $\mathbf{A}$ it holds

$$\phi \vdash \psi \quad\Longrightarrow\quad \langle\phi, G\rangle \geq \langle\psi, G\rangle.$$

However, even if ϕ and ψ are sentences and $\langle\phi, \cdot\,\rangle \geq \langle\psi, \cdot\,\rangle$ on finite λ-structures, this does not imply in general that $\phi \vdash \psi$: let θ be a sentence with only infinite models and let ϕ be a sentence with only finite models. On finite λ-structures it holds $\langle\phi \vee \theta, \cdot\,\rangle = \langle\phi, \cdot\,\rangle$ although $\phi \vee \theta \nvdash \phi$ (as witnessed by an infinite model of θ).

Nevertheless, inequalities between Stone pairing that are valid for finite λ-structures will of course still hold at the limit. For instance, for $\phi_1, \phi_2 \in \mathrm{FO}_1(\lambda)$, for $\zeta \in \mathrm{FO}_2(\lambda)$, and for $a, b \in \mathbb{N}$ define the first-order sentence $B(a, b, \phi_1, \phi_2, \zeta)$ expressing that for every vertex x such that $\phi_1(x)$ holds there exist at least a vertices y such that $\phi_2(y) \wedge \zeta(x, y)$ holds and that for every vertex y such that $\phi_2(x)$ holds there exist at most b vertices x such that $\phi_1(x) \wedge \zeta(x, y)$ holds. Then it is easily checked that for every finite λ-structure $\mathbf{A}$ it holds

$$\mathbf{A} \models B(a, b, \phi_1, \phi_2, \zeta) \quad\Longrightarrow\quad a\langle\phi_1, \mathbf{A}\rangle \leq b\langle\phi_2, \mathbf{A}\rangle.$$

For example, if a finite directed graph is such that every arc connects a vertex with out-degree 2 to a vertex with in-degree 1, it is clear that the probability that a random vertex has out-degree 2 is half the probability that a random vertex has in-degree 1.

Now we come to important twist and the basic of our approach. The Stone pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ can be considered from both sides: On the right side the functions of type $\langle\phi, \cdot\,\rangle$ are a generalization of the homomorphism density functions [10]:

$$t(F, G) = \frac{|\hom(F, G)|}{|G|^{|F|}}$$

(these functions correspond to $\langle\phi, G\rangle$ for Boolean conjunctive queries ϕ and a graph G). Also the density function used in [6] to measure the probability that the ball of radius r rooted at a random vertex as a given isomorphism type may be expressed as a function $\langle\phi, \cdot\rangle$. We follow here, in a sense, a dual approach: from the left side we consider for fixed $\mathbf{A}$ the function $\langle\,\cdot\,,\mathbf{A}\rangle$, which is an additive function on $\mathcal{B}(\mathrm{FO}(\lambda))$ with the following properties:

- $\langle\,\cdot\,,\mathbf{A}\rangle \geq 0$ and $\langle 1,\mathbf{A}\rangle = 1$;
- $\langle\sigma\cdot\phi,\mathbf{A}\rangle = \langle\phi,\rangle$ for every $\sigma \in S_\omega$;
- if $\mathrm{Fv}(\phi)\cap\mathrm{Fv}(\psi) = \emptyset$, then $\langle\phi\wedge\psi,\mathbf{A}\rangle = \langle\phi,\mathbf{A}\rangle\ \langle\psi,\mathbf{A}\rangle$.

Thus $\langle\,\cdot\,,\mathbf{A}\rangle$ is, for a given $\mathbf{A}$, an operator on the class of first-order formulas.

We now can apply Lemma 3 to derive a representation by means of a regular measure on a Stone space. The fine structure and interplay of additive functions, Boolean functions, and dual spaces can be used effectively if we consider finite λ-structures as probability spaces as we did when we considered finite λ-structures as a particular case of Borel models.

Theorem 5. *Let λ be a signature, let $\mathcal{B}(\mathrm{FO}(\lambda))$ be the Lindenbaum-Tarski algebra of $\mathrm{FO}(\lambda)$, let $S(\mathcal{B}(\mathrm{FO}(\lambda)))$ be the associated Stone space, and let $\mathrm{rca}(S(\mathcal{B}(\mathrm{FO}(\lambda))))$ be the Banach space of the regular countably additive measure on $S(\mathcal{B}(\mathrm{FO}(\lambda)))$. Then:*

(1) There is a mapping from the class of λ-modeling to $\mathrm{rca}(S(\mathcal{B}(\mathrm{FO}(\lambda))))$, which maps a modeling $\mathbf{A}$ to the unique regular measure $\mu_{\mathbf{A}}$ such that for every $\phi\in\mathrm{FO}(\lambda)$ it holds

$$\langle\phi,\mathbf{A}\rangle = \int_{S(\mathcal{B}(\mathrm{FO}(\lambda)))} \mathbf{1}_{K(\phi)}\,\mathrm{d}\mu_{\mathbf{A}},$$

where $\mathbf{1}_{K(\phi)}$ is the indicator function of $K(\phi)$ in $S(\mathcal{B}(\mathrm{FO}(\lambda)))$. Moreover, this mapping is injective of finite λ-structures.

(2) A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of finite λ-structures is FO-convergent if and only if the sequence $(\mu_{\mathbf{A}_n})_{n\in\mathbb{N}}$ is weakly converging in $\mathrm{rca}(S(\mathcal{B}(\mathrm{FO}(\lambda))))$;

(3) If $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is an FO-convergent sequence of finite λ-structures then the weak limit μ of $(\mu_{\mathbf{A}_n})_{n\in\mathbb{N}}$ is such that for every $\phi\in\mathrm{FO}(\lambda)$ it holds

$$\lim_{n\to\infty}\langle\phi,\mathbf{A}_n\rangle = \int_{S(\mathcal{B}(\mathrm{FO}(\lambda)))} \mathbf{1}_{K(\phi)}\,\mathrm{d}\mu.$$

Proof. The proof follows from Lemma 3, considering the additive functions $\langle\,\cdot\,,\mathbf{A}\rangle$.

Let $\mathbf{A}$ be a finite λ-structure. As $\mu_{\mathbf{A}}$ allows to recover the complete theory of $\mathbf{A}$ and as $\mathbf{A}$ is finite, the mapping $\mathbf{A}\mapsto\mu_{\mathbf{A}}$ is injective. □ □

It will be convenient to sometimes consider fragments of $\mathrm{FO}(\lambda)$ to define a weaker notion of convergence.

Definition 7 (X-convergence)**.** Let X be a fragment of $\mathrm{FO}(\lambda)$. A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of finite λ-structures is *X-convergent* if $\langle\phi,\mathbf{A}_n\rangle$ is convergent for every $\phi\in X$.

In this context, we can extend Theorem 5.

Theorem 6. *Let λ be a signature, and let X be a fragment of $\mathrm{FO}(\lambda)$ defining a Boolean algebra $\mathcal{B}(X)\subseteq\mathcal{B}(\mathrm{FO}(\lambda))$. Let $S(\mathcal{B}(X))$ be the associated Stone space,*

and let $\mathrm{rca}(S(\mathcal{B}(X)))$ *be the Banach space of the regular countably additive measure on* $S(\mathcal{B}(X))$*. Then:*

(1) *The canonical injection* $\iota^X : \mathcal{B}(X) \to \mathcal{B}(\mathrm{FO}(\lambda))$ *defines by duality a continuous projection* $p^X : S(\mathcal{B}(\mathrm{FO}(\lambda))) \to S(\mathcal{B}(X))$*; The pushforward* $p^X_* \mu_{\mathbf{A}}$ *of the measure* $\mu_{\mathbf{A}}$ *associated to a modeling* $\mathbf{A}$ *(see Theorem 5) is the unique regular measure on* $S(\mathcal{B}(X))$ *such that:*

$$\langle \phi, \mathbf{A} \rangle = \int_{S(\mathcal{B}(X))} \mathbf{1}_{K(\phi)} \,\mathrm{d}p^X_* \mu_{\mathbf{A}},$$

where $\mathbf{1}_{K(\phi)}$ *is the indicator function of* $K(\phi)$ *in* $S(\mathcal{B}(X))$*.*

(2) *A sequence* $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *of finite* λ*-structures is* X*-convergent if and only if the sequence* $(p^X_* \mu_{\mathbf{A}_n})_{n\in\mathbb{N}}$ *is weakly converging in* $\mathrm{rca}(S(\mathcal{B}(X)))$*;*

(3) *If* $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *is an* X*-convergent sequence of finite* λ*-structures then the weak limit* μ *of* $(p^X_* \mu_{\mathbf{A}_n})_{n\in\mathbb{N}}$ *is such that for every* $\phi \in X$ *it holds*

$$\lim_{n\to\infty} \langle \phi, \mathbf{A}_n \rangle = \int_{S(\mathcal{B}(X))} \mathbf{1}_{K(\phi)} \,\mathrm{d}\mu.$$

Proof. If X is closed under conjunction, disjunction and negation, thus defining a Boolean algebra $\mathcal{B}(X)$, then the inclusion of X in $\mathrm{FO}(\lambda)$ translates as a canonical injection ι from $\mathcal{B}(X)$ to $\mathcal{B}(\mathrm{FO}(\lambda))$. By Stone duality, the injection ι corresponds to a continuous projection $p : S(\mathcal{B}(\mathrm{FO}(\lambda))) \to S(\mathcal{B}(X))$. As every measurable function, this continuous projection also transports measures by pushforward: the projection p transfers the measure μ on $S(\mathcal{B}(\mathrm{FO}(\lambda)))$ to $S(\mathcal{B}(X))$ as the pushforward measure $p_*\mu$ defined by the identity $p_*\mu(Y) = \mu(p^{-1}(Y))$, which holds for every measurable subset Y of $S(\mathcal{B}(X))$.

The proof follows from Lemma 3, considering the additive functions $\langle \cdot, \mathbf{A} \rangle$. □

□

We can also consider a notion of convergence restricted to λ-structures satisfying a fixed axiom.

Theorem 7. *Let* λ *be a signature, and let* X *be a fragment of* $\mathrm{FO}(\lambda)$ *defining a Boolean algebra* $\mathcal{B}(X) \subseteq \mathcal{B}(\mathrm{FO}(\lambda))$*. Let* $S(\mathcal{B}(X))$ *be the associated Stone space, and let* $\mathrm{rca}(S(\mathcal{B}(X)))$ *be the Banach space of the regular countably additive measure on* $S(\mathcal{B}(X))$*.*

Let $\mathcal{C}$ *be a basic elementary class defined by a single axiom* $\Psi \in X \cap \mathrm{FO}_0$*, and let* $\mathcal{I}_\Psi$ *be the principal ideal of* $\mathcal{B}(X)$ *generated by* $\neg\Psi$*.*

Then:

(1) *The Boolean algebra obtained by taking the quotient of* X *equivalence modulo* Ψ *is the quotient Boolean algebra* $\mathcal{B}(X, \Psi) = \mathcal{B}(X)/\mathcal{I}_\Psi$*. Then* $S(\mathcal{B}(X, \Psi))$ *is homeomorphic to the clopen subspace* $K(\Psi)$ *of* $S(\mathcal{B}(X))$*.*

If $\mathbf{A} \in \mathcal{C}$ *is a finite* λ*-structure then the support of the measure* $p^X_* \mu_{\mathbf{A}}$ *associated to* $\mathbf{A}$ *(see Theorem 6) is included in* $K(\Psi)$ *and for every* $\phi \in X$ *it holds*

$$\langle \phi, \mathbf{A} \rangle = \int_{K(\Psi)} \mathbf{1}_{K(\phi)} \,\mathrm{d}p^X_* \mu_{\mathbf{A}}.$$

(2) *A sequence* $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *of finite* λ*-structures of* $\mathcal{C}$ *is* X*-convergent if and only if the sequence* $(p^X_* \mu_{\mathbf{A}_n})_{n\in\mathbb{N}}$ *is weakly converging in* $\mathrm{rca}(S(\mathcal{B}(X, \Psi)))$*;*

(3) *If* $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *is an* X*-convergent sequence of finite* λ*-structures in* $\mathcal{C}$ *then the weak limit* μ *of* $(p^X_* \mu_{\mathbf{A}_n})_{n\in\mathbb{N}}$ *is such that for every* $\phi \in X$ *it holds*

$$\lim_{n\to\infty} \langle \phi, \mathbf{A}_n \rangle = \int_{K(\Psi)} \mathbf{1}_{K(\phi)} \,\mathrm{d}\mu.$$

Proof. The quotient algebra $\mathcal{B}(X,\Psi) = \mathcal{B}(X)/\mathcal{I}_\Psi$ is isomorphic to the sub-Boolean algebra $\mathcal{B}'$ of $\mathcal{B}$ of all (equivalence classes of) formulas $\phi \wedge \Psi$ for $\phi \in X$. To this isomorphism corresponds by duality the identification of $S(\mathcal{B}(X,\Psi))$ with the clopen subspace $K(\Psi)$ of $S(\mathcal{B}(X))$. □ □

The situation expressed by these theorems is summarized in the following diagram.

$$\begin{tikzcd}
\mathcal{B}(\mathrm{FO}(\lambda)) \arrow[d, leftrightarrow] & \mathcal{B}(X) \arrow[l, "\text{canonical injection}"'] \arrow[d, leftrightarrow] & \mathcal{B}' \arrow[l, "\text{inclusion}"'] \arrow[r, leftrightarrow, "\textit{isomorphism}"] \arrow[d, leftrightarrow] & \mathcal{B}(X,\Psi) \arrow[d, leftrightarrow] \\
S(\mathcal{B}(\mathrm{FO}(\lambda))) \arrow[r, "\text{projection } p^X"] & S(\mathcal{B}(X)) & K(\Psi) \arrow[l, "\text{inclusion}"'] \arrow[r, leftrightarrow, "\text{homeomorphism}"] & S(\mathcal{B}(X,\Psi))
\end{tikzcd}$$

$$\mu \xrightarrow{\text{pushforward}} p^X_*\,\mu \xrightarrow{\text{restriction}} p^X_*\,\mu$$

We shall now examine more closely X-convergence for fragments X of $\mathrm{FO}(\lambda)$ which are relevant to examples and limits previously studied and consider in particular the structure of the Stone spaces $S(\mathcal{B}(X))$.

4. Convergence, Old and New

As we have seen above, there are many ways how to say that a sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of finite λ-structures is convergent. As we considered λ-structures defined with a countable signature λ, the Boolean algebra $\mathcal{B}(\mathrm{FO}(\lambda))$ is countable. It follows that the Stone space $S(\mathcal{B}(\mathrm{FO}(\lambda)))$ is a Polish space thus (with the Borel σ-algebra) it is a standard Borel space. Hence every probability distribution turns $S(\mathcal{B}(\mathrm{FO}(\lambda)))$ into a standard probability space. However, the fine structure of $S(\mathcal{B}(\mathrm{FO}(\lambda)))$ is complex and we have no simple description of this space.

FO-convergence is of course the most restrictive notion of convergence and it seems (at least on the first glance) that this is perhaps too much to ask, as we may encounter many particular difficulties and specific cases. But we shall exhibit later classes for which FO-convergence is captured — for special basic elementary classes of structures — by X-convergence for a small fragment X of FO.

At this time it is natural to ask whether one can consider fragments that are not sub-Boolean algebras of $\mathrm{FO}(\mathcal{L})$ and still have a description of the limit of a converging sequence as a probability measure on a nice measurable space. There is obviously a case where this is possible: when the convergence of $\langle\phi,\mathbf{A}_n\rangle$ for every ϕ in a fragment X implies the convergence of $\langle\psi,\mathbf{A}_n\rangle$ for every ψ in the minimum Boolean algebra containing X. We prove now that this is for instance the case when X is a fragment closed under conjunction.

We shall need the following preliminary lemma:

Lemma 4. *Let $X \subseteq B$ be closed by $\wedge$ and such that* X generates B *(i.e. such that $B[X] = B$).*

Then $\{\mathbf{1}_b :\ b \in X\} \cup \{\mathbf{1}\}$ (where $\mathbf{1}$ is the constant function with value 1*) includes a basis of the vector space $V(B)$ generated by the whole set $\{\mathbf{1}_b :\ b \in B\}$.*

Proof. Let $b \in B$. As X generates B there exist $b_1,\dots,b_k \in X$ and a Boolean function F such that $b = F(b_1,\dots,b_k)$. As $\mathbf{1}_{x\wedge y} = \mathbf{1}_x\,\mathbf{1}_y$ and $\mathbf{1}_{\neg x} = \mathbf{1} - \mathbf{1}_x$ there exists a polynomial P_F such that $\mathbf{1}_b = P_F(\mathbf{1}_{b_1},\dots,\mathbf{1}_{b_k})$. For $I \subseteq [k]$, the monomial $\prod_{i\in I}\mathbf{1}_{b_i}$ rewrites as $\mathbf{1}_{b_I}$ where $b_I = \bigwedge_{i\in I} b_i$. It follows that $\mathbf{1}_b$ is a linear

combination of the functions $\mathbf{1}_{b_I}$ ($I \subseteq [k]$) which belong to X if $I \neq \emptyset$ (as X is closed under $\wedge$ operation) and equal $\mathbf{1}$, otherwise. □ □

Lemma 5. *Let X be a fragment of* FO(λ) *closed under (finite) conjunction — thus defining a* meet semilattice *of* $\mathcal{B}(\mathrm{FO}(\lambda))$ *— and let $\mathcal{B}(X)$ be the sub-Boolean algebra of* $\mathcal{B}(\mathrm{FO}(\lambda))$ *generated by X. Let $\overline{X}$ be the fragment of* FO(λ) *consisting of all formulas with equivalence class in $\mathcal{B}(X)$.*

Then X-convergence is equivalent to $\overline{X}$-convergence.

Proof. Let $\Psi \in \overline{X}$. According to Lemma 4, there exist $\phi_1, \dots, \phi_k \in X$ and $\alpha_0, \alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that

$$\mathbf{1}_\Psi = \alpha_0 \mathbf{1} + \sum_{i=1}^{k} \alpha_i \mathbf{1}_{\phi_i}.$$

Let $\mathbf{A}$ be a λ-structure, let $\Omega = S(\mathcal{B}(X))$ and let $\mu_{\mathbf{A}} \in \mathrm{rca}(\Omega)$ be the associated measure. Then

$$\langle \Psi, \mathbf{A}\rangle = \int_\Omega \mathbf{1}_\Psi \,\mathrm{d}\mu_{\mathbf{A}} = \int_\Omega \big(\alpha_0 \mathbf{1} + \sum_{i=1}^{k} \alpha_i \mathbf{1}_{\phi_i}\big)\,\mathrm{d}\mu_G = \alpha_0 + \sum_{i=1}^{k} \alpha_i \langle \phi_i, \mathbf{A}\rangle.$$

It follows that if $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is an X-convergent sequence, the sequence $(\langle \psi, \mathbf{A}_n\rangle)_{n\in\mathbb{N}}$ converges for every $\psi \in \overline{X}$, that is $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is $\overline{X}$-convergent. □ □

We now will demonstrate the expressive power of X-convergence by relating it to the main types of convergence of graphs studied previously:

(1) the notion of *dense graph limit* [9, 46];
(2) the notion of *bounded degree graph limit* [6, 3];
(3) the notion of *elementary limit* derived from two important results in first-order logic, namely Gödel's completeness theorem and the compactness theorem.

These standard notions of graph limits, which have inspired this work, correspond to special fragments of FO(λ), where γ is the signature of graphs. In the remaining of this section, we shall only consider undirected graphs, thus we shall omit to precise their signature in the notations as well as the axiom defining the basic elementary class of undirected graphs.

4.1. **L-convergence and** QF**-convergence.** Recall that a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is *L-convergent* if

$$t(F, G_n) = \frac{\hom(F, G_n)}{|G_n|^{|F|}}$$

converges for every fixed (connected) graph F, where $\hom(F, G)$ denotes the number of homomorphisms of F to G [46, 11, 12].

It is a classical observation that homomorphisms between finite structures can be expressed by Boolean conjunctive queries [14]. We denote by HOM the fragment of FO consisting of formulas formed by conjunction of atoms. For instance, the formula

$$(x_1 \sim x_2) \wedge (x_2 \sim x_3) \wedge (x_3 \sim x_4) \wedge (x_4 \sim x_5) \wedge (x_5 \sim x_1)$$

belongs to HOM and it expresses that $(x_1, x_2, x_3, x_4, x_5)$ form a homomorphic image of C_5. Generally, to a finite graph F we associate the canonical formula $\phi_F \in$ HOM defined by

$$\phi_F := \bigwedge_{ij\in E(F)} (x_i \sim x_j).$$

Then, for every graph G it holds

$$\langle \phi_F, G\rangle = \frac{\hom(F,G)}{|G|^{|F|}} = t(F,G).$$

Thus L-convergence is equivalent to HOM-convergence. According to Lemma 5, HOM-convergence is equivalent to $\overline{\mathrm{HOM}}$-convergent. It is easy to see that $\overline{\mathrm{HOM}}$ is the fragment QF^- of quantifier free formulas that do not use equality. We prove now that HOM-convergence is actually equivalent to QF-convergence, where QF is the fragment of all quantifier free formulas. Note that QF is a proper fragment of $\mathrm{FO}^{\mathrm{local}}$.

Theorem 8. *Let (G_n) be a sequence of finite graphs such that $\lim_{n\to\infty}|G_n| = \infty$. Then the following conditions are equivalent:*

(1) the sequence (G_n) is L-convergent;
(2) the sequence (G_n) is QF^--convergent;
(3) the sequence (G_n) is QF-convergent;

Proof. As L-convergence is equivalent to HOM-convergence and as $\mathrm{HOM} \subset \mathrm{QF}^- \subset \mathrm{QF}$, it is sufficient to prove that L-convergence implies QF-convergence.

Assume (G_n) is L-convergent. The inclusion-exclusion principle implies that for every finite graph F the density of induced subgraphs isomorphic to F converges too. Define

$$\mathrm{dens}(F, G_n) = \frac{(\#F \subseteq_i G_n)}{|G_n|^{|F|}}.$$

Then $\mathrm{dens}(F, G_n)$ is a converging sequence for each F.

Let θ be a quantifier-free formula with $\mathrm{Fv}(\theta) \subseteq [p]$. We first consider all possible cases of equalities between the free variables. For a partition $\mathcal{P} = (I_1, \dots, I_k)$ of $[p]$, we define $|\mathcal{P}| = k$ and $s_{\mathcal{P}}(i) = \min I_i$ (for $1 \le i \le |\mathcal{P}|$). Consider the formula

$$\zeta_{\mathcal{P}} := \bigwedge_{i=1}^{|\mathcal{P}|}\Bigl(\bigwedge_{j\in I_i}(x_j = x_{s_{\mathcal{P}}(i)}) \ \wedge \bigwedge_{j=i+1}^{|\mathcal{P}|}(x_{s_{\mathcal{P}}(j)} \neq x_{s_{\mathcal{P}}(i)})\Bigr).$$

Then θ is logically equivalent to

$$(\bigwedge_{i\neq j}(x_i \neq x_j) \wedge \theta) \ \vee \bigvee_{\mathcal{P}:|\mathcal{P}|<p} \zeta_{\mathcal{P}} \wedge \theta_{\mathcal{P}}(x_{s_{\mathcal{P}}(1)}, \dots, x_{s_{\mathcal{P}}(|\mathcal{P}|)}).$$

Note that all the formulas in the disjunction are mutually exclusive. Also $\bigwedge_{i\neq j}(x_i \neq x_j) \wedge \theta$ may be expressed as a disjunction of mutually exclusive terms:

$$\bigwedge_{i\neq j}(x_i \neq x_j) \wedge \theta = \bigvee_{F\in\mathcal{F}} \theta'_F,$$

where $\mathcal{F}$ is a finite family of finite graphs F and where $G \models \theta'_F(v_1, \dots, v_p)$ if and only if the mapping $i \mapsto v_i$ is an isomorphism from F to $G[v_1, \dots, v_p]$.

It follows that for every graph G it holds:

$$\begin{aligned}
\langle\theta, G\rangle &= \sum_{F\in\mathcal{F}}\langle\theta'_F, G\rangle + \sum_{\mathcal{P}:|\mathcal{P}|<p}\langle\zeta_{\mathcal{P}}\wedge\theta_{\mathcal{P}}(x_{s_{\mathcal{P}}(1)},\dots,x_{s_{\mathcal{P}}(|\mathcal{P}|)}), G\rangle\\
&= \sum_{F\in\mathcal{F}}\langle\theta'_F, G\rangle + \sum_{\mathcal{P}:|\mathcal{P}|<p}|G|^{|\mathcal{P}|-p}\langle\theta_{\mathcal{P}}, G\rangle\\
&= \sum_{F\in\mathcal{F}}\frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\frac{|\{(v_1,\dots,v_p):\ G\models\theta'_F(v_{\sigma(1)},\dots,v_{\sigma(p)})\}|}{|G|^p} + O(|G|^{-1})\\
&= \sum_{F\in\mathcal{F}}\frac{\mathrm{Aut}(F)}{p!}\,\mathrm{dens}(F,G) + O(|G|^{-1}).
\end{aligned}$$

Thus $\langle\theta, G_n\rangle$ converge for every quantifier-free formula θ. Hence (G_n) is QF-convergent. □ □

Notice that the condition that $\lim_{n\to\infty}|G_n|$ is necessary as witnessed by the sequence (G_n) where G_n is K_1 if n is odd and $2K_1$ if n is even. The sequence is obviously L-convergent, but not QF convergent as witnessed by the formula $\phi(x,y): x\neq y$, which has density 0 in K_1 and 1/2 in K_2.

Remark 2. The Stone space of the fragment QF^- has a simple description. Indeed, a homomorphism $h:\mathcal{B}(\mathrm{QF}^-)\to\mathbf{2}$ is determined by its values on the formulas $x_i\sim x_j$ and any mapping from this subset of formulas to $\mathbf{2}$ extends (in a unique way) to a homomorphism of $\mathcal{B}(\mathrm{QF}^-)$ to $\mathbf{2}$. Thus the points of $S(\mathcal{B}(\mathrm{QF}^-))$ can be identified with the mappings from $\binom{\mathbb{N}}{2}$ to $\{0,1\}$ that is to the graphs on $\mathbb{N}$. Hence the considered measures μ are probability measures of graphs on $\mathbb{N}$ that have the property that they are invariant under the natural action of S_ω on $\mathbb{N}$. Such random graphs on $\mathbb{N}$ are called *infinite exchangeable random graphs*. For more on infinite exchangeable random graphs and graph limits, see e.g. [4, 16].

4.2. **BS-convergence and** $\mathrm{FO}^{\mathrm{local}}$**-convergence.** The class of graphs with maximum degree at most D (for some integer D) received much attention. Specifically, the notion of *local weak convergence* of bounded degree graphs was introduced in [6], which is called here *BS-convergence*:

A *rooted graph* is a pair (G,o), where $o\in V(G)$. An *isomorphism* of rooted graph $\phi:(G,o)\to(G',o')$ is an isomorphism of the underlying graphs which satisfies $\phi(o)=o'$. Let $D\in\mathbb{N}$. Let $\mathcal{G}_D$ denote the collection of all isomorphism classes of connected rooted graphs with maximal degree at most D. For the sake of simplicity, we denote elements of $\mathcal{G}_D$ simply as graphs. For $(G,o)\in\mathcal{G}_D$ and $r\geq 0$ let $B_G(o,r)$ denote the subgraph of G spanned by the vertices at distance at most r from o. If $(G,o),(G',o')\in\mathcal{G}_D$ and r is the largest integer such that $(B_G(o,r),o)$ is rooted-graph isomorphic to $(B_{G'}(o',r),o')$, then set $\rho((G,o),(G',o'))=1/r$, say. Also take $\rho((G,o),(G,o))=0$. Then ρ is metric on $\mathcal{G}_D$. Let $\mathfrak{M}_D$ denote the space of all probability measures on $\mathcal{G}_D$ that are measurable with respect to the Borel σ-field of ρ. Then $\mathfrak{M}_D$ is endowed with the topology of weak convergence, and is compact in this topology.

A sequence $(G_n)_{n\in\mathbb{N}}$ of finite connected graphs with maximum degree at most D is *BS-convergent* if, for every integer r and every rooted connected graph (F,o) with maximum degree at most D the following limit exists:

$$\lim_{n\to\infty}\frac{|\{v: B_{G_n}(v,r)\cong(F,o)\}|}{|G_n|}.$$

This notion of limits leads to the definition of a limit object as a probability measure on $\mathcal{G}_D$ [6].

To relate BS-convergence to X-convergence, we shall consider the fragment of local formulas:

Let $r \in \mathbb{N}$. A formula $\phi \in \mathrm{FO}_p$ is *r-local* if, for every graph G and every $v_1, \dots, v_p \in G^p$ it holds

$$G \models \phi(v_1, \dots, v_p) \qquad \iff \qquad G[N_r(v_1, \dots, v_p)] \models \phi(v_1, \dots, v_p),$$

where $G[N_r(v_1, \dots, v_p)]$ denotes the subgraph of G induced by all the vertices at (graph) distance at most r from one of $v_1, \dots, v_p$ in G.

A formula ϕ is *local* if it is r-local for some $r \in \mathbb{N}$; the fragment $\mathrm{FO}^{\mathrm{local}}$ is the set of all local formulas in FO. Notice that if ϕ_1 and ϕ_2 are local formulas, so are $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$ and $\neg\phi_1$. It follows that the quotient of $\mathrm{FO}^{\mathrm{local}}$ by the relation of logical equivalence defines a sub-Boolean algebra $\mathcal{B}(\mathrm{FO}^{\mathrm{local}})$ of $\mathcal{B}(\mathrm{FO})$. For $p \in \mathbb{N}$ we further define $\mathrm{FO}_p^{\mathrm{local}} = \mathrm{FO}^{\mathrm{local}} \cap \mathrm{FO}_p$.

Theorem 9. *Let (G_n) be a sequence of finite graphs with maximum degree d, with $\lim_{n\to\infty} |G_n| = \infty$.*

Then the following properties are equivalent:

(1) the sequence $(G_n)_{n\in\mathbb{N}}$ is BS-convergent;
(2) the sequence $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent;
(3) the sequence $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent.

Proof. If $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent, it is $\mathrm{FO}_1^{\mathrm{local}}$-convergent;

If $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent then it is BS-convergent as for any finite rooted graph (F, o), testing whether the the ball of radius r centered at a vertex x is isomorphic to (F, o) can be formulated by a local first order formula.

Assume $(G_n)_{n\in\mathbb{N}}$ is BS-convergent. As we consider graphs with maximum degree d, there are only finitely many isomorphism types for the balls of radius r centered at a vertex. It follows that any local formula $\xi(x)$ with a single variable can be expressed as the conjunction of a finite number of (mutually exclusive) formulas $\xi_{(F,o)}(x)$, which in turn correspond to subgraph testing. It follows that BS-convergence implies $\mathrm{FO}_1^{\mathrm{local}}$-convergence.

Assume $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent and let $\phi \in \mathrm{FO}_p^{\mathrm{local}}$ be an r-local formula. Let $\mathcal{F}_\phi$ be the set of all p-tuples $((F_1, f_1), \dots, (F_p, f_p))$ of rooted connected graphs with maximum degree at most d and radius (from the root) at most r such that $\bigcup_i F_i \models \phi(f_1, \dots, f_p)$.

Then, for every graph G the sets

$$\Omega_\phi(G) = \{(v_1, \dots, v_p) : \ G \models \phi(v_1, \dots, v_p)\}$$

and

$$\biguplus_{((F_1,f_1),\dots,(F_p,f_p))\in\mathcal{F}_\phi} \prod_{i=1}^{p} \{v : \ G \models \theta_{(F_i,f_i)}(v)\}$$

differ by at most $O(|G|^{p-1})$ elements. Indeed, according to the definition of an r-local formula, the p-tuples $(x_1, \dots, x_p)$ belonging to exactly one of these sets are such that there exists $1 \le i < j \le p$ such that $\mathrm{dist}(x_i, x_j) \le 2r$.

It follows that

$$\langle \phi, G\rangle = \Big(\sum_{((F_i,f_i))_{1\le i\le p}\in\mathcal{F}_\phi} \prod_{i=1}^{p} \langle \theta_{(F_i,f_i)}, G\rangle \Big) + O(|G|^{-1}).$$

It follows that $\mathrm{FO}_1^{\mathrm{local}}$-convergence (hence BS-convergence) implies full $\mathrm{FO}^{\mathrm{local}}$-convergence. □ □

Remark 3. According to this proposition and Theorem 7, the BS-limit of a sequence of graphs with maximum degree at most D corresponds to a probability measure on $S(\mathcal{B}(\mathrm{FO}_1^{\mathrm{local}}))$ whose support is include in the clopen set $K(\zeta_D)$, where ζ_D is the sentence expressing that the maximum degree is at most D. The Boolean algebra $\mathcal{B}(\mathrm{FO}_1^{\mathrm{local}})$ is isomorphic to the Boolean algebra defined by the fragment $X \subset \mathrm{FO}_0(\lambda_1)$ of sentences for rooted graphs that are local with respect to the root (here, λ_1 denotes the signature of graphs augmented by one symbol of constant). According to this locality, any two countable rooted graphs (G_1, r_1) and (G_2, r_2), the trace of the complete theories of (G_1, r_1) and (G_2, r_2) on X are the same if and only if the (rooted) connected component (G_1', r_1) of (G_1, r_1) containing the root r_1 is elementary equivalent to the (rooted) connected component (G_2', r_2) of (G_2, r_2) containing the root r_2. As isomorphism and elementary equivalence are equivalent for countable connected graphs with bounded degrees (see Lemma 6) it is easily checked that $K_X(\zeta_D)$ is homeomorphic to $\mathcal{G}_D$. Hence our setting (while based on a very different and dual approach) leads essentially to the same limit object as [6] for BS-convergent sequences.

4.3. **Elementary-convergence and FO_0-convergence.** We already mentioned that FO_0-convergence is nothing but elementary convergence. Elementary convergence is implicitly part of the classical model theory. Although we only consider graphs here, the definition and results indeed generalize to general λ-structures We now reword the notion of elementary convergence:

A sequence $(G_n)_{n\in\mathbb{N}}$ is *elementarily convergent* if, for every sentence $\phi \in \mathrm{FO}_0$, there exists a integer N such that either all the graphs G_n ($n \geq N$) satisfy ϕ or none of them do.

Of course, the limit object (as a graph) is not unique in general and formally, the limit of an elementarily convergent sequence of graphs is an elementary class defined by a complete theory.

Elementary convergence is also the backbone of all the X-convergences we consider in this paper. The FO_0-convergence is induced by an easy ultrametric defined on equivalence classes of elementarily equivalent graphs. Precisely, two (finite or infinite) graphs G_1, G_2 are *elementarily equivalent* (denoted $G_1 \equiv G_2$) if, for every sentence ϕ it holds

$$G_1 \models \phi \iff G_2 \models \phi.$$

In other words, two graphs are elementarily equivalent if they satisfy the same sentences.

A weaker (parametrized) notion of equivalence will be crucial: two graphs G_1, G_2 are *k-elementarily equivalent* (denoted $G_1 \equiv^k G_2$) if, for every sentence ϕ with quantifier rank at most k it holds $G_1 \models \phi \iff G_2 \models \phi$.

It is easily checked that for every two graphs G_1, G_2 it holds:

$$G_1 \equiv G_2 \qquad \iff \qquad (\forall k \in \mathbb{N})\ G_1 \equiv^k G_2.$$

For every fixed $k \in \mathbb{N}$, checking whether two graphs G_1 and G_2 are k-elementarily equivalent can be done using the so-called Ehrenfeucht-Fraïssé game.

From the notion of k-elementary equivalence naturally derives a pseudometric $\mathrm{dist}_0(G_1, G_2)$:

$$\mathrm{dist}_0(G_1, G_2) = \begin{cases} 0 & \text{if } G_1 \equiv G_2 \\ \min\{2^{-\mathrm{qrank}(\phi)} : (G_1 \models \phi) \wedge (G_2 \models \neg\phi)\} & \text{otherwise} \end{cases}$$

Proposition 1. *The metric space of countable graphs (up to elementary equivalence) with ultrametric* dist_0 *is compact.*

Proof. This is a direct consequence of the compactness theorem for first-order logic (a theory has a model if and only if every finite subset of it has a model) and of the downward Löwenheim-Skolem theorem (if a theory has a model and the language is countable then the theory has a countable model). □ □

Note that not every countable graph is (up to elementary equivalence) the limit of a sequence of finite graphs. A graph G that is a limit of a sequence finite graphs is said to have the *finite model property*, as such a graph is characterized by the property that every finite set of sentences satisfied by G has a finite model (what does not imply that G is elementarily equivalent to a finite graph).

Example 1. A *ray* is not an elementary limit of finite graphs as it contains exactly one vertex of degree 1 and all the other vertices have degree 2, what can be expressed in first-order logic but is satisfied by no finite graph. However, the union of two rays is an elementary limit from the sequence $(P_n)_{n\in\mathbb{N}}$ of paths of order n.

Although two finite graphs are elementary equivalent if and only if they are isomorphic, this property does not holds in general for countable graphs. For instance, the union of a ray and a line is elementarily equivalent to a ray. However we shall make use of the equivalence of isomorphisms and elementary equivalences for rooted connected countable locally finite graphs, which we prove now for completeness.

Lemma 6. *Let* (G, r) *and* (G', r') *be two rooted connected countable graphs.*

If G *is locally finite then* $(G, r) \equiv (G', r')$ *if and only if* (G, r) *and* (G', r') *are isomorphic.*

Proof. If two rooted graphs are isomorphic they are obviously elementarily equivalent. Assume that (G, r) and (G', r') are elementarily equivalent. Enumerate the vertices of G in a way that distance to the root is not decreasing. Using n-back-and-forth equivalence (for all $n \in \mathbb{N}$), one builds a tree of partial isomorphisms of the subgraphs induced by the n first vertices, where ancestor relation is restriction. This tree is infinite and has only finite degrees. Hence, by Kőnig's lemma, it contains an infinite path. It is easily checked that it defines an isomorphism from (G, r) to (G', r') as these graphs are connected. □ □

Fragments of FO_0 allow to define convergence notions, which are weaker that elementary convergence. The hierarchy of the convergence schemes defined by sub-algebras of $\mathcal{B}(\mathrm{FO}_0)$ is as strict as one could expect. Precisely, if $X \subset Y$ are two sub-algebras of $\mathcal{B}(\mathrm{FO}_0)$ then Y-convergence is strictly stronger than X-convergence — meaning that there exists graph sequences that are X-convergent but not Y-convergent — if and only if there exists a sentence $\phi \in Y$ such that for every sentence $\psi \in X$, there exists a (finite) graph G disproving $\phi \leftrightarrow \psi$.

We shall see that the special case of elementary convergent sequences is of particular importance. Indeed, every limit measure is a Dirac measure concentrated on a single point of $S(\mathcal{B}(\mathrm{FO}_0))$. This point is the complete theory of the elementary limit of the considered sequence. This limit can be represented by a finite or countable graph. As FO-convergence (and any FO_p-convergence) implies FO_0-convergence, the support of a limit measure μ corresponding to an FO_p-convergent sequence (or to an FO-convergent sequence) is such that $\mathrm{Supp}(\mu)$ projects to a single point of $S(\mathcal{B}(\mathrm{FO}_0))$.

Finally, let us remark that all the results of this section can be readily formulated and proved for λ-structures.

5. Combining Fragments

5.1. **The FO_p Hierarchy.** When we consider FO_p-convergence of finite λ-structures for finite a signature λ, the space $S(\mathcal{B}(\mathrm{FO}_p(\lambda)))$ can be given the following ultrametric dist_p (compatible with the topology of $S(\mathcal{B}(\mathrm{FO}_p(\lambda)))$): Let $T_1, T_2 \in S(\mathcal{B}(\mathrm{FO}_p(\lambda)))$ (where the points of $S(\mathcal{B}(\mathrm{FO}_p(\lambda)))$ are identified with ultrafilters on $\mathcal{B}(\mathrm{FO}_p(\lambda))$). Then

$$\mathrm{dist}_p(T_1, T_2) = \begin{cases} 0 & \text{if } T_1 = T_2 \\ 2^{-\min\{\mathrm{qrank}(\phi):\ \phi \in T_1 \setminus T_2\}} & \text{otherwise} \end{cases}$$

This ultrametric has several other nice properties:

- actions of S_p on $S(\mathcal{B}(\mathrm{FO}_p(\lambda)))$ are isometries:

$$\forall \sigma \in S_p\ \forall T_1, T_2 \in S(\mathcal{B}(\mathrm{FO}_p(\lambda))) \quad \mathrm{dist}_p(\sigma \cdot T_1, \sigma \cdot T_2) = \mathrm{dist}_p(T_1, T_2);$$

- projections π_p are contractions:

$$\forall q \geq p\ \forall T_1, T_2 \in S(\mathcal{B}(\mathrm{FO}_q(\lambda))) \quad \mathrm{dist}_p(\pi_p(T_1), \pi_p(T_2)) \leq \mathrm{dist}_q(T_1, T_2);$$

We prove that there is a natural isometric embedding $\eta_p : S(\mathcal{B}(\mathrm{FO}_p(\lambda))) \to S(\mathcal{B}(\mathrm{FO}(\lambda)))$. This may be seen as follows: for an ultrafilter $X \in S(\mathcal{B}(\mathrm{FO}_p(\lambda)))$, consider the filter X^+ on $\mathcal{B}(\mathrm{FO}(\lambda))$ generated by X and all the formulas $x_i = x_{i+1}$ (for $i \geq p$). This filter is an ultrafilter: for every sentence $\phi \in \mathrm{FO}(\lambda)$, let $\widetilde{\phi}$ be the sentence obtained from ϕ by replacing each free occurrence of a variable x_q with $q > p$ by x_p. It is clear that ϕ and $\widetilde{\phi}$ are equivalent modulo the theory $T_p = \{(x_i = x_{i+1}) :\ i \geq p\}$. As either $\widetilde{\phi}$ or $\neg\widetilde{\phi}$ belongs to X, either ϕ or $\neg\phi$ belongs to $\eta_p(X)$. Moreover, we deduce easily from the fact that $\widetilde{\phi}$ and ϕ have the same quantifier rank that if $q \geq p$ then $\pi_q \circ \eta_p$ is an isometry. Finally, let us note that $\pi_p \circ \eta_p$ is the identity of $S(\mathcal{B}(\mathrm{FO}_p(\lambda)))$.

Let λ_p be the signature λ augmented by p symbols of constants $c_1, \dots, c_p$. There is a natural isomorphism of Boolean algebras $\nu_p : \mathrm{FO}_p(\lambda) \to \mathrm{FO}_0(\lambda_p)$, which replaces the free occurrences of the variables $x_1, \dots, x_p$ in a formula $\phi \in \mathrm{FO}_p$ by the corresponding symbols of constants $c_1, \dots, c_p$, so that it holds, for every modeling $\mathbf{A}$, for every $\phi \in \mathrm{FO}_p$ and every $v_1, \dots, v_p \in A$:

$$\mathbf{A} \models \phi(v_1, \dots, v_p) \quad \iff \quad (\mathbf{A}, v_1, \dots, v_p) \models \nu_p(\phi).$$

This mapping induces an isometric isomorphism of the metric spaces $(S(\mathcal{B}(\mathrm{FO}_p(\lambda))), \mathrm{dist}_p)$ and $(S(\mathcal{B}(\mathrm{FO}_0(\lambda_p))), \mathrm{dist}_0)$. Note that the Stone space $S(\mathcal{B}(\mathrm{FO}_0(\lambda_p)))$ associated to the Boolean algebra $\mathcal{B}(\mathrm{FO}_0(\lambda_p))$ is the space of all complete theories of λ_p-structures. In particular, points of $S(\mathcal{B}(\mathrm{FO}_p(\lambda))$ can be represented (up to elementary equivalence) by countable λ-structures with p special points. All these transformations may seem routine but they need to be carefully formulated and checked.

We can test whether the distance dist_p of two theories T and T' is smaller than 2^{-n} by means of an Ehrenfeucht-Fraïssé game: Let $\nu_p(T) = \{\nu_p(\phi) : \phi \in T\}$ and, similarly, let $\nu_p(T') = \{\nu_p(\phi) : \phi \in T'\}$. Let $(\mathbf{A}, v_1, \dots, v_p)$ be a model of T and let $(\mathbf{A}', v_1', \dots, v_p')$ ba a model of T'. Then it holds

$$\mathrm{dist}_p(T, T') < 2^{-n} \quad \iff \quad (\mathbf{A}, v_1, \dots, v_p) \equiv^n (\mathbf{A}', v_1', \dots, v_p').$$

Recall that the n-rounds *Ehrenfeucht-Fraïssé game* on two λ-structures $\mathbf{A}$ and $\mathbf{A}'$, denoted $\mathrm{EF}(\mathbf{A}, \mathbf{A}', n)$ is the perfect information game with two players — the Spoiler and the Duplicator — defined as follows: The game has n rounds and each round has two parts. At each round, the Spoiler first chooses one of $\mathbf{A}$ and $\mathbf{A}'$ and accordingly selects either a vertex $x \in A$ or a vertex $y \in A'$. Then, the Duplicator

selects a vertex in the other λ-structure. At the end of the n rounds, n vertices have been selected from each structure: $x_1,\dots,x_n$ in A and $y_1,\dots,y_n$ in A' (x_i and y_i corresponding to vertices x and y selected during the ith round). The Duplicator wins if the substructure induced by the selected vertices are order-isomorphic (i.e. $x_i \mapsto y_i$ is an isomorphism of $\mathbf{A}[\{x_1,\dots,x_n\}]$ and $\mathbf{A}'[\{y_1,\dots,y_n\}]$). As there are no hidden moves and no draws, one of the two players has a winning strategy, and we say that that player wins $\mathrm{EF}(\mathbf{A},\mathbf{A}',n)$. The main property of this game is the following equivalence, due to Fraïssé [27, 28] and Ehrenfeucht [22]: The duplicator wins $\mathrm{EF}(\mathbf{A},\mathbf{A}',n)$ if and only if $\mathbf{A}\equiv^n\mathbf{A}'$. In our context this translates to the following equivalence:

$$\mathrm{dist}_p(T,T') < 2^{-n} \iff \text{Duplicator wins } \mathrm{EF}((\mathbf{A},v_1,\dots,v_p),(\mathbf{A}',v_1',\dots,v_p'),n).$$

As $\mathrm{FO}_0\subset\mathrm{FO}_1\subset\dots\subset\mathrm{FO}_p\subset\mathrm{FO}_{p+1}\subset\dots\subset\mathrm{FO}=\bigcup_i\mathrm{FO}_i$, the fragments FO form a hierarchy of more and more restrictive notions of convergence. In particular, FO_{p+1}-convergence implies FO_p-convergence and FO-convergence is equivalent to FO_p for all p. If a sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is FO_p-convergent then for every $q\le p$ the FO_q-limit of $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is a measure $\mu_q\in\mathrm{rca}(S(\mathcal{B}(\mathrm{FO}_q)))$, which is the pushforward of μ_p by the projection π_q (more precisely, by the restriction of π_q to $S(\mathcal{B}(\mathrm{FO}_p))$):

$$\mu_q=(\pi_q)_*(\mu_p).$$

5.2. $\mathrm{FO}^{\mathrm{local}}$ **and Locality.** FO-convergence can be reduced to the conjunction of elementary convergence and $\mathrm{FO}^{\mathrm{local}}$-convergence, which we call *local convergence.* This is a consequence of Gaifman locality theorem, which we recall now.

Theorem 10 (Gaifman locality theorem [30])**.** *For every first-order formula $\phi(x_1,\dots,x_n)$ there exist integers t and r such that ϕ is equivalent to a Boolean combination of t-local formulas $\xi_s(x_{i_1},\dots,x_{i_s})$ and sentences of the form*

$$\exists y_1\dots\exists y_m\Big(\bigwedge_{1\le i<j\le m}\mathrm{dist}(y_i,y_j)>2r\wedge\bigwedge_{1\le i\le m}\psi(y_i)\Big)\tag{3}$$

where ψ is r-local. Furthermore, we can choose

$$r\le 7^{\mathrm{qrank}(\phi)-1},\ t\le(7^{\mathrm{qrank}(\phi)-1}-1)/2,\ m\le n+\mathrm{qrank}(\phi),$$

and, if ϕ is a sentence, only sentences (3) *occur in the Boolean combination. Moreover, these sentences can be chosen with quantifier rank at most $q(\mathrm{qrank}(\phi))$, for some fixed function q.*

From this theorem and the following folklore technical result will follow the claimed decomposition of FO-convergence into elementary and local convergence.

Lemma 7. *Let B be a Boolean algebra, let A_1 and A_2 be sub-Boolean algebras of B, and let $b\in B[A_1\cup A_2]$ be a Boolean combination of elements from A_1 and A_2. Then b can be written as*

$$b=\bigvee_{i\in I}x_i\wedge y_i,$$

where I is finite, $x_i\in A_1$, $y_i\in A_2$, and for every $i\neq j$ in I it holds $(x_i\wedge y_i)\wedge(x_j\wedge y_j)=0$.

Proof. Let $b=F(u_1,\dots,u_a,v_1,\dots,v_b)$ with $u_i\in A_1$ $(1\le i\le a)$ and $v_j\in A_2$ $(1\le j\le b)$ where F is a Boolean combination. By using iteratively Shannon's expansion, we can write F as

$$F(u_1,\dots,u_a,v_1,\dots,v_b)=\bigvee_{(X_1,X_2,Y_1,Y_2)\in\mathcal{F}}(\bigwedge_{i\in X_1}u_i\wedge\bigwedge_{i\in X_2}\neg u_i\wedge\bigwedge_{j\in Y_1}v_j\wedge\bigwedge_{j\in Y_2}\neg v_j),$$

where $\mathcal{F}$ is a subset of the quadruples (X_1, X_2, Y_1, Y_2) such that (X_1, X_2) is a partition of $[a]$ and (Y_1, Y_2) is a partition of $[b]$. For a quadruple $Q = (X_1, X_2, Y_1, Y_2)$, define $x_Q = \bigwedge_{i \in X_1} u_i \wedge \bigwedge_{i \in X_2} \neg u_i$ and $y_Q = \bigwedge_{j \in Y_1} v_j \wedge \bigwedge_{j \in Y_2} \neg v_j$. Then for every $Q \in \mathcal{F}$ it holds $x_Q \in A_1, y_Q \in A_2$, for every $Q \neq Q' \in \mathcal{F}$ it holds $x_Q \wedge y_Q \wedge x_{Q'} \wedge y_{Q'} = 0$, and we have $b = \bigvee_{Q \in \mathcal{F}} x_Q \wedge y_Q$. □ □

Theorem 11. *Let $(\mathbf{A}_n)$ be a sequence of finite λ-structures. Then $(\mathbf{A}_n)$ is* FO*-convergent if and only if it is both* $\mathrm{FO}^{\mathrm{local}}$*-convergent and* FO_0*-convergent. Precisely, $(\mathbf{A}_n)$ is* FO_p*-convergent if and only if it is both* $\mathrm{FO}_p^{\mathrm{local}}$*-convergent and* FO_0*-convergent.*

Proof. Assume $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is both $\mathrm{FO}_p^{\mathrm{local}}$-convergent and FO_0-convergent and let $\phi \in \mathrm{FO}_p$. According to Theorem 10, there exist integers t and r such that ϕ is equivalent to a Boolean combination of t-local formula $\xi(x_{i_1}, \dots, x_{i_s})$ and of sentences. As both $\mathrm{FO}^{\mathrm{local}}$ and FO_0 define a sub-Boolean algebra of $\mathcal{B}(\mathrm{FO})$, according to Lemma 7, ϕ can be written as $\bigvee_{i\in I} \psi_i \wedge \theta_i$, where I is finite, $\psi_i \in \mathrm{FO}^{\mathrm{local}}$, $\theta_i \in \mathrm{FO}_0$, and $\psi_i \wedge \theta_i \wedge \psi_j \wedge \theta_j = 0$ if $i \neq j$. Thus for every finite λ-structure $\mathbf{A}$ it holds

$$\langle \phi, \mathbf{A} \rangle = \sum_{i\in I} \langle \psi_i \wedge \theta_i, \mathbf{A} \rangle.$$

As $\langle\,\cdot\,, \mathbf{A}\rangle$ is additive and $\langle \theta_i, \mathbf{A}\rangle \in \{0,1\}$ we have $\langle \psi_i \wedge \theta_i, \mathbf{A}\rangle = \langle \psi_i, \mathbf{A}\rangle\, \langle \theta_i, \mathbf{A}\rangle$. Hence

$$\langle \phi, \mathbf{A} \rangle = \sum_{i\in I} \langle \psi_i, \mathbf{A} \rangle\, \langle \theta_i, \mathbf{A} \rangle.$$

Thus if $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is both $\mathrm{FO}_p^{\mathrm{local}}$-convergent and FO_0-convergent then $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is FO_p-convergent. □ □

Similarly that points of $S(\mathcal{B}(\mathrm{FO}_p(\lambda))$ can be represented (up to elementary equivalence) by countable λ-structures with p special points, points of $S(\mathcal{B}(\mathrm{FO}_p^{\mathrm{local}}(\lambda))$ can be represented by countable λ-structures with p special points such that every connected component contains at least one special point. In particular, points of $S(\mathcal{B}(\mathrm{FO}_1^{\mathrm{local}}(\lambda))$ can be represented by rooted connected countable λ-structures.

Also, the structure of an $\mathrm{FO}_2^{\mathrm{local}}$-limit of graphs can be outlined by considering that points of $S(\mathcal{B}(\mathrm{FO}_2^{\mathrm{local}}))$ as countable graphs with two special vertices c_1 and c_2, such that every connected component contains at least one of c_1 and c_2. Let μ_2 be the limit probability measure on $S(\mathcal{B}(\mathrm{FO}_2^{\mathrm{local}}))$ for an $\mathrm{FO}_2^{\mathrm{local}}$-convergent sequence $(G_n)_{n\in\mathbb{N}}$, let π_1 be the standard projection of $S(\mathcal{B}(\mathrm{FO}_2^{\mathrm{local}}))$ into $S(\mathcal{B}(\mathrm{FO}_1^{\mathrm{local}}))$, and let μ_1 be the pushforward of μ_2 by π_1. We construct a measurable graph $\hat{G}$ as follows: the vertex set of $\hat{G}$ is the support $\mathrm{Supp}(\mu_1)$ of μ_1. Two vertices x and y of $\hat{G}$ are adjacent if there exists $x' \in \pi_1^{-1}(x)$ and $y' \in \pi_1^{-1}(y)$ such that (considered as ultrafilters of $\mathcal{B}(\mathrm{FO}_2^{\mathrm{local}})$) it holds:

- $x_1 \sim x_2$ belongs to both x' and y',
- the transposition $\tau_{1,2}$ exchanges x' and y' (i.e. $y' = \tau_{1,2} \cdot x'$).

The vertex set of $\hat{G}$ is of course endowed with a structure of a probability space (as a measurable subspace of $S(\mathcal{B}(\mathrm{FO}_1^{\mathrm{local}}))$ equipped with the probability measure μ_1). In the case of bounded degree graphs, the obtained graph $\hat{G}$ is the *graph of graphs* introduced in [44]. Notice that this graph may have loops. An example of such a graph is shown Fig. 1.

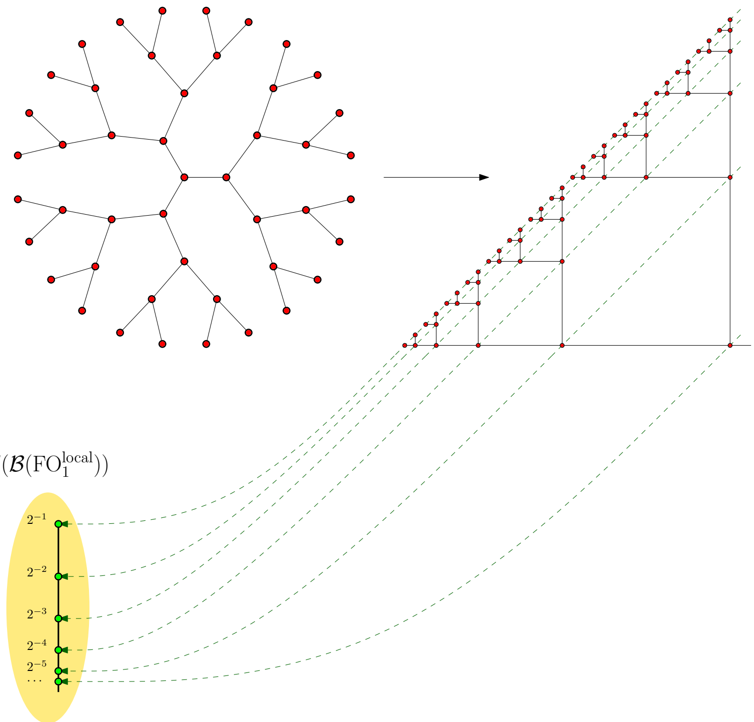

FIGURE 1. An outline of the local limit of a sequence of trees

5.3. **Sequences with Homogeneous Elementary Limit.** Elementary convergence is an important aspect of FO-convergence and we shall see that in several contexts, FO-convergence can be reduced to the conjunction and elementary convergence of X-convergence (for some suitable fragment X).

In some special cases, the limit (as a countable structure) will be unique. This means that some particular complete theories have exactly one countable model (up to isomorphism). Such complete theories are called *ω-categorical*. Several properties are known to be equivalent to ω-categoricity. For instance, for a complete theory T the following statements are equivalent:

- T is ω-categorical;
- for every every $p \in \mathbb{N}$, the Stone space $S(\mathcal{B}(\mathrm{FO}_p(\lambda), T))$ is finite (see Fig. 2);
- every countable model $\mathbf{A}$ of T has an *oligomorphic* automorphism group, what means that for every $n \in \mathbb{N}$, A^n has finitely many orbits under the action of $\mathrm{Aut}(\mathbf{A})$.

A theory T is said to have *quantifier elimination* if, for every formula $\phi \in \mathrm{FO}_p(\lambda)$ there exists $\widetilde{\phi} \in \mathrm{QF}_p(\lambda)$ such that $T \models \phi \leftrightarrow \widetilde{\phi}$. If a theory has quantifier elimination then it is ω-categorical. Indeed, for every p, there exists only finitely many quantifier free formulas with p free variables hence (up to equivalence modulo T) only finitely many formulas with p free variables. The unique countable model of a complete theory T with quantifier elimination is *ultra-homogeneous*, what means that every partial isomorphism of finite induced substructures extends as a full automorphism. In the context of relational structures, the property of having a countable ultra-homogeneous model is equivalent to the property of having quantifier elimination. We provide a proof of this folklore result (in the context of graphs) in order to illustrate these notions.

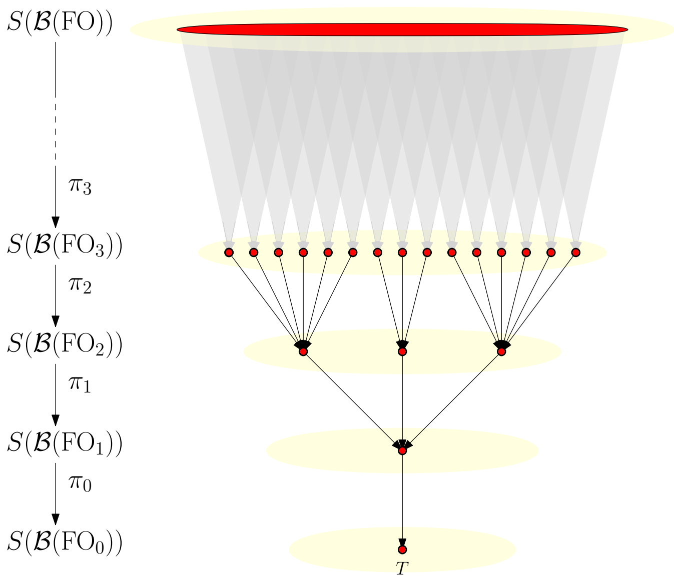

FIGURE 2. Ultrafilters projecting to an ω-categorical theory

Lemma 8. *Let T be a complete theory (of graphs) with no finite model.*

Then T has quantifier elimination if and only if some (equivalently, every) countable model of T is ultra-homogeneous.

Proof. Assume that T has an ultra-homogeneous countable model G. Let $(a_1,\dots,a_p)$, $(b_1,\dots,b_p)$ be p-tuples of vertices of G. Assume that $a_i \mapsto b_i$ is an isomorphism between $G[a_1,\dots,a_p]$ and $G[b_1,\dots,b_p]$. Then, as G is ultra-homogeneous, there exists an automorphism f of G such that $f(a_i) = b_i$ for every $1 \leq i \leq p$. As the satisfaction of a first-order formula is invariant by the action of the automorphism group, for every formula $\phi \in \mathrm{FO}_p$ it holds

$$G \models \phi(a_1,\dots,a_p) \quad\iff\quad G \models \phi(b_1,\dots,b_p).$$

Consider a maximal set $\mathcal{F}$ of p-tuples $(v_1,\dots,v_p)$ of G such that $G \models \phi(v_1,\dots,v_p)$ and no two p-tuples induce isomorphic (ordered) induced subgraphs. Obviously $|\mathcal{F}| = 2^{O(p^2)}$ is finite. Moreover, each p-tuple $\vec{v} = (v_1,\dots,v_p)$ defines a quantifier free formula $\eta_{\vec{v}}$ with p free variables such that $G \models \eta_{\vec{v}}(x_1,\dots,x_p)$ if and only if $x_i \mapsto v_i$ is an isomorphism between $G[x_1,\dots,x_p]$ and $G[v_1,\dots,v_p]$. Hence it holds:

$$G \models \phi \leftrightarrow \bigvee_{\vec{v}\in\mathcal{F}} \eta_{\vec{v}}.$$

In other words, ϕ is equivalent (modulo T) to the quantifier free formula $\widetilde{\phi} = \bigvee_{\vec{v}\in\mathcal{F}} \eta_{\vec{v}}$, that is: T has quantifier elimination.

Conversely, assume that T has quantifier elimination. As notice above, T is ω-categorical thus has a unique countable model. Assume $(a_1,\dots,a_p)$ and $(b_1,\dots,b_p)$ are p-tuples of vertices such that $f : a_i \mapsto b_i$ is a partial isomorphism. Assume that f does not extend into an automorphism of G. Let $(a_1,\dots,a_q)$ be a tuple of vertices of G of maximal length such that there exists $b_{p+1},\dots,b_q$ such that $a_i \mapsto b_i$ is a partial isomorphism. Let a_{q+1} be a vertex distinct from $a_1,\dots,a_q$. Let

$\phi(x_1,\dots,x_q)$ be the formula

$$\bigwedge_{a_i\sim a_j}(x_i\sim x_j)\wedge\bigwedge_{\neg(a_i\sim a_j)}\neg(x_i\sim x_j)\wedge\bigwedge_{1\leq i\leq q}\neg(x_i=x_j)$$
$$\wedge\,(\exists y)\left(\bigwedge_{a_i\sim a_{q+1}}(x_i\sim y)\wedge\bigwedge_{\neg(a_i\sim a_{q+1})}\neg(x_i\sim y)\wedge\bigwedge_{1\leq i\leq q}\neg(x_i=y)\right)$$

As T has quantifier elimination, there exists a quantifier free formula $\widetilde{\phi}$ such that $T\models\phi\leftrightarrow\widetilde{\phi}$. As $G\models\phi(a_1,\dots,a_q)$ (witnessed by a_{q+1}) it holds $G\models\widetilde{\phi}(a_1,\dots,a_q)$ hence $G\models\widetilde{\phi}(b_1,\dots,b_q)$ (as $a_i\mapsto b_i, 1\leq i\leq q$ is a partial isomorphism) thus $G\models\phi(b_1,\dots,b_q)$. It follows that there exists b_{q+1} such that $a_i\mapsto b_i, 1\leq i\leq q+1$ is a partial isomorphism, contradicting the maximality of $(a_1,\dots,a_q)$. □ □

When a sequence of graphs is elementarily convergent to an ultra-homogeneous graph (i.e. to a complete theory with quantifier elimination), we shall prove that FO-convergence reduces to QF-convergence. This later mode of convergence is of particular interest as it is equivalent to L-convergence, as we first prove.

We now prove that for sequences of graphs elementarily convergent to ultra-homogeneous graphs, the properties of FO-convergence and QF-convergence are equivalent.

Lemma 9. *Let $(G_n)_{n\in\mathbb{N}}$ be sequence of graphs that converges elementarily to some ultra-homogeneous graph $\hat{G}$. Then the following properties are equivalent:*

- *the sequence $(G_n)_{n\in\mathbb{N}}$ is* FO*-convergent;*
- *the sequence $(G_n)_{n\in\mathbb{N}}$ is* QF*-convergent.*
- *the sequence $(G_n)_{n\in\mathbb{N}}$ is L-convergent.*

Proof. As FO-convergence implies QF-convergence we only have to prove the opposite direction. Assume that the sequence $(G_n)_{n\in\mathbb{N}}$ is QF-convergent. According to Lemma 8, for every formula $\phi\in\mathrm{FO}_p$ there exists a quantifier free formula $\widetilde{\phi}\in\mathrm{QF}_p$ such that $\hat{G}\models\phi\leftrightarrow\widetilde{\phi}$ (i.e. $\mathrm{Th}(\hat{G})$ has quantifier elimination). As $\hat{G}$ is an elementary limit of the sequence $(G_n)_{n\in\mathbb{N}}$ there exists N such that for every $n\geq N$ it holds $G_n\models\phi\leftrightarrow\widetilde{\phi}$. It follows that for every $n\geq N$ it holds $\langle\phi,G_n\rangle=\langle\widetilde{\phi},G_n\rangle$ hence $\lim_{n\to\infty}\langle\phi,G_n\rangle$ exists. Thus the sequence $(G_n)_{n\in\mathbb{N}}$ is FO-convergent. □ □

There are not so many countable ultra-homogeneous graphs.

Theorem 12 (Lachlan and Woodrow [40])**.** *Every infinite countable ultrahomogeneous undirected graph is isomorphic to one of the following:*

- *the disjoint union of m complete graphs of size n, where $m,n\leq\omega$ and at least one of m or n is ω, (or the complement of it);*
- *the generic graph for the class of all countable graphs not containing K_n for a given $n\geq 3$ (or the complement of it).*
- *the Rado graph R (the generic graph for the class of all countable graphs).*

Among them, the Rado graph R is characterized by the *extension property*: for every finite disjoint subsets of vertices A and B of R there exists a vertex z of $R-A-B$ such that z is adjacent to every vertex in A and to no vertex in B. The Rado graph R will be of great importance for us, because of the following property.

Lemma 10 (Edrös, Rényi [26])**.** *Let $0<\delta<1$, let $p_{i,j}\in[\delta,1-\delta]$ for $i,j\in\mathbb{N}$ and let G be the random countable graph with vertex set $\mathbb{N}$ that is such that — denoting $E_{i,j}$ the event that i is adjacent to j — the events $E_{i,j}$ are independent and the*

probability of $E_{i,j}$ is $p_{i,j}$. Then, with probability one G has the extension property (hence is isomorphic to the Rado graph).

In particular, for $0 < p < 1$, let G_n be a random graph with n vertices where two vertices are adjacent with probability p, independently for each pair of vertices. Then with probability 1 the resulting graph sequence $(G_n)_{n\in\mathbb{N}}$ will converge elementarily to the Rado graph.

We now related more precisely the extension property with quantifier elimination.

Definition 8. Let $k \in \mathbb{N}$. A graph G has the *k-extension property* if, for every disjoint subsets of vertices A, B of G with size k there exists a vertex z not in $A \cup B$ that is adjacent to every vertex in A and to no vertex in B. In other words, G has the k-extension property if G satisfies the sentence Υ_k below:

$$(\forall x_1, \dots, x_{2k}) \quad \Big(\bigwedge_{1\le i<j\le 2k} \neg(x_i = x_j) \\ \rightarrow \quad (\exists z) \bigwedge_{i=1}^{2k} \neg(x_i = z) \wedge \bigwedge_{i=1}^{k} (x_i \sim z) \wedge \bigwedge_{i=k+1}^{2k} \neg(x_i \sim z) \Big)$$

Lemma 11. *Let G be a graph and let p, r be integers.*

If G has the $(p+r)$-extension property then every formula ϕ with p free variables and quantifier rank r is equivalent, in G, with a quantifier free formula.

Proof. Let ϕ be a formula with p free variables and quantifier rank r. Let $(a_1, \dots, a_p)$ and $(b_1, \dots, b_p)$ be two p-tuples of vertices of G such that $a_i \mapsto b_i$ is a partial isomorphism. The $(p + r)$-extension properties allows to easily play a r-turns back-and-forth game between $(G, a_1, \dots, a_p)$ and $(G, b_1, \dots, b_p)$, thus proving that $(G, a_1, \dots, a_p)$ and $(G, b_1, \dots, b_p)$ are r-equivalent. It follows that $G \models \phi(a_1, \dots, a_p)$ if and only if $G \models \phi(b_1, \dots, b_p)$. Following the lines of Lemma 8, we deduce that there exists a quantifier free formula $\widetilde{\phi}$ such that $G \models \phi \leftrightarrow \widetilde{\phi}$. □ □

Lemma 12. *Let $1/2 > \delta > 0$. Assume that for every positive integer $n \ge 2$ and every $1 \le i < j \le n$, $p_{n,i,j} \in [\delta, 1-\delta]$. Assume that for each $n \in \mathbb{N}$, G_n is a random graph on $[f(n)]$ where $f(n) \ge n$, and where i and j are adjacent with probability $p_{n,i,j}$ (all these events being independent). Then the sequence $(G_n)_{n\in\mathbb{N}}$ almost surely converges elementarily to the Rado graph.*

Proof. Let $p \in \mathbb{N}$ and let $\alpha = \delta(1-\delta)$. The probability that $G_n \models \Upsilon_p$ is at least $1 - (1-\alpha^p)^{f(n)}$. It follows that for $N \in \mathbb{N}$ the probability that all the graphs G_n ($n \ge N$) satisfy Υ_p is at least $1 - \alpha^{-p}(1-\alpha^p)^{f(N)}$. According to Borel-Cantelli lemma, the probability that G_n does not satisfy Υ_p infinitely many is zero. As this holds for every integer p, it follows that, with high probability, every elementarily converging subsequence of $(G_n)_{n\in\mathbb{N}}$ converges to the Rado graph hence, with high probability, $(G_n)_{n\in\mathbb{N}}$ converges elementarily to the Rado graph. □ □

Thus we get:

Theorem 13. *Let $0 < p < 1$ and let $G_n \in G(n,p)$ be independent random graphs with edge probability p. Then $(G_n)_{n\in\mathbb{N}}$ is almost surely* FO*-convergent.*

Proof. This is an immediate consequence of Lemma 9, Lemma 12 and the easy fact that $(G_n)_{n\in\mathbb{N}}$ is almost surely QF-convergent. □ □

Theorem 14. *For every $\phi \in \mathrm{FO}_p$ there exists a polynomial $P_\phi \in \mathbb{Z}[X_1, \dots, X_{\binom{p}{2}}]$ such that for every sequence $(G_n)_{n\in\mathbb{N}}$ of finite graphs that converges elementarily to the Rado graph the following holds:*

If $(G_n)_{n\in\mathbb{N}}$ is L-convergent to some graphon W then

$$\lim_{n\to\infty} \langle \phi, G_n\rangle = \int \cdots \int P_\phi((W_{i,j}(x_i, x_j))_{1\leq i<j\leq p})\, \mathrm{d}x_1 \dots \mathrm{d}x_p.$$

Proof. Assume the sequence $(G_n)_{n\in\mathbb{N}}$ is elementarily convergent to the Rado graph and that it is L-convergent to some graphon W.

According to Lemma 8, there exists a quantifier free formula $\widetilde{\phi}$ such that

$$G \models (\forall x_1 \dots x_p)\ \phi(x_1, \dots, x_p) \leftrightarrow \widetilde{\phi}(x_1, \dots, x_p)$$

(hence $\Omega_\phi(G) = \Omega_{\widetilde{\phi}}(G)$) holds when G is the Rado graph. As $(G_n)_{n\in\mathbb{N}}$ is elementarily convergent to the Rado graph, this sentence holds for all but finitely many graphs G_n. Thus for all but finitely many G_n it holds $\langle\phi, G_n\rangle = \langle\widetilde{\phi}, G_n\rangle$. Moreover, according to Lemma 9, the sequence $(G_n)_{n\in\mathbb{N}}$ is FO-convergent and thus it holds

$$\lim_{n\to\infty} \langle \phi, G_n\rangle = \lim_{n\to\infty} \langle \widetilde{\phi}, G_n\rangle.$$

By using inclusion/exclusion argument and the general form of the density of homomorphisms of fixed target graphs to a graphon we deduce that there exists a polynomial $P_\phi \in \mathbb{Z}[X_1, \dots, X_{\binom{p}{2}}]$ (which depends only on ϕ) such that

$$\lim_{n\to\infty} \langle \widetilde{\phi}, G_n\rangle = \int \cdots \int P_\phi((W_{i,j}(x_i, x_j))_{1\leq i<j\leq p})\, \mathrm{d}x_1 \dots \mathrm{d}x_p.$$

The theorem follows. □

Although elementary convergence to Rado graph seems quite a natural assumption for graphs which are neither too sparse nor too dense, elementary convergence to other ultra-homogeneous graphs may be problematic.

Example 2. Cherlin [15] posed the problem whether there is a finite k-saturated triangle-free graph, for each $k \in \mathbb{N}$, where a triangle free graph is called *k-saturated* if for every set S of at most k vertices, and for every independent subset T of S, there exists a vertex adjacent to each vertex of T and to no vertex of $S - T$. In other words, Cherlin asks whether the generic countable triangle-free graph has the finite model property, that is if it is an elementary limit of a sequence of finite graphs.

It is possible to extend Lemma 9 to sequences of graph having a non ultra-homogeneous elementary limit if we restrict FO to a smaller fragment.

Example 3. A graph G is *IH-Homogeneous* [13] if every partial finite isomorphism extends into an endomorphism. Let PP be the fragment of FO that consists into *primitive positive* formulas, that is formulas formed using adjacency, equality, conjunctions and existential quantification only, and let BA(PP) be the minimum sub-Boolean algebra of FO containing PP.

Following the lines of Lemma 9 and using Theorem 8 and Lemma 5, one proves that if a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ converges elementarily to some IH-homogeneous infinite countable graph then $(G_n)_{n\in\mathbb{N}}$ is BA(PP)-convergent if and only if it is QF-convergent.

5.4. FO-**convergence of Graphs with Bounded Maximum Degree.** We now consider how full FO-convergence differs to BS-convergence for sequence of graphs with maximum degree at most D. As a corollary of Theorems 11 and 9 we have:

Theorem 15. *A sequence (G_n) of finite graphs with maximum degree at most d such that $\lim_{n\to\infty}|G_n|=\infty$ is* FO*-convergent if and only if it is both BS-convergent and elementarily convergent.*

6. Non-Standard Intermezzo

We show that limit objects which are close to modelings can be obtained by a non-standard approach. Our goal is Theorem 18. The use of a non-standard approach in the area of graph and hypergraph limits was pioneered by Elek and Szegedy and we shall follow closely their paper [24].

We first recall the ultraproduct construction. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of graphs and let U be a non-principal ultrafilter. Let $\widetilde{G}=\prod_{i\in\mathbb{N}}G_i$ and let $\sim$ be the equivalence relation on $\widetilde{V}$ defined by $(x_n)\sim(y_n)$ if $\{n:x_n=y_n\}\in U$. Then the *ultraproduct* of the graphs G_n is the quotient of $\widetilde{G}$ by $\sim$, and it is denoted $\prod_U G_i$. Two vertices $[x]$ and $[y]$ are adjacent if $\{n:x_n \text{ is adjacent to } y_n\}\in U$. (Notice that this construction easily extends to relational structures.)

The fundamental theorem of ultraproducts proved by Łoś makes ultraproducts particularly useful in model theory. We express it now in the particular case of graphs indexed by $\mathbb{N}$ but its general statement concerns structures indexed by a set I and the ultraproduct constructed by considering an ultrafilter U over I.

Theorem 16 ([43])**.** *For each formula $\phi\in\mathrm{FO}_n$ and each $f_1,\dots,f_n\in\prod_i G_i$ we have*

$$\prod_U G_i\models\phi([f_1],\dots,[f_n])\quad\textit{iff}\quad\{i:\ G_i\models\phi(f_1(i),\dots,f_n(i))\}\in U.$$

Note that if (G_i) is elementary-convergent, then $\prod_U G_i$ is an elementary limit of the sequence: for every sentence ϕ, according to Theorem 16, we have

$$\prod_U G_i\models\phi\quad\iff\quad\{i:\ G_i\models\phi\}\in U.$$

Remark 4. It is easily checked that a (possibly infinite) λ-structure G is an elementary limit of a sequence $(G_n)_{n\in\mathbb{N}}$ of finite λ-structure if and only if every first-order sentence (in $\mathrm{FO}_0(\lambda)$) which is true in G has a finite model. This, in turn, is equivalent (see [65], Lemma 1) to the property that there exists a (countable) set $\{G_i:i\in I\}$ of finite λ-structures and an ultrafilter U on I such that G is elementarily equivalent to the ultraproduct $\prod_U G_i$. In this sense, ultraproducts of finite structures are natural limit objects for elementarily convergent sequences of finite structures.

A measure ν extending the normalised counting measures ν_i of G_i is then obtained via the Loeb measure construction. We denote by $\mathcal{P}(G_i)$ the Boolean algebra of the subsets of vertices of G_i, with the normalized measure $\nu_i(A)=\frac{|A|}{|G_i|}$. We define $\mathcal{P}=\prod_i\mathcal{P}(G_i)/I$, where I is the ideal of the elements $\{A_i\}_{i\in\mathbb{N}}$ such that $\{i:\ A_i=\emptyset\}\in U$. We have

$$[x]\in[A]\quad\text{iff}\quad\{i:\ x_i\in A_i\}\in U.$$

These sets form a Boolean algebra over $\prod_U G_i$. Recall that the ultralimit $\lim_U a_n$ defined for every $(a_n)_{n\in\mathbb{N}}\in\ell^\infty(\mathbb{N})$ is such that for every $\epsilon>0$ we have

$$\{i:\ a_i \in [\lim_U a_n - \epsilon\,;\ \lim_U a_n + \epsilon]\} \in U.$$

Define

$$\nu([A]) = \lim_U \nu_i(A_i).$$

Then $\nu : \mathcal{P} \to \mathbb{R}$ is a finitely additive measure. Remark that, according to Hahn-Kolmogorov theorem, proving that ν extends to a countably additive measure amounts to prove that for every sequence $([A^n])$ of disjoint elements of $\mathcal{P}$ such that $\bigcup_n [A^n] \in \mathcal{P}$ it holds $\nu(\bigcup_n [A^n]) = \sum_n \nu([A^n])$.

A subset $N \subseteq \prod_U G_i$ is a *nullset* if for every $\epsilon > 0$ there exists $[A^\epsilon] \in \mathcal{P}$ such that $N \subseteq [A^\epsilon]$ and $\nu([A^\epsilon]) < \epsilon$. The set of nullsets is denoted by $\mathcal{N}$. A set $B \subseteq \prod_U G_i$ is *measurable* if there exists $\widetilde{B} \in \mathcal{P}$ such that $B \Delta \widetilde{B} \in \mathcal{N}$.

The following theorem is proved in [24]:

Theorem 17. *The measurable sets form a σ-algebra B_U and $\nu(B) = \nu(\widetilde{B})$ defines a probability measure on B_U.*

Notice that this construction extends to the case where to each G_i is associated a probability measure ν_i. Then the limit measure ν is non-atomic if and only if the following technical condition holds: for every $\epsilon > 0$ and for every $(A_n) \in \prod G_n$, if for U-almost all n it holds $\nu_n(A_n) \geq \epsilon$ then there exists $\delta > 0$ and $(B_n) \in \prod G_n$ such that for U-almost all n it holds $B_n \subseteq A_n$ and $\min(\nu_n(B_n), \nu_n(A_n \setminus B_n)) \geq \delta$. This obviously holds if ν_n is a normalized counting measure and $\lim_U |G_n| = \infty$. Let ν be the limit measure.

Let $f_i : G_i \to [-d; d]$ be real functions, where $d > 0$. One can define $f : \prod_U G_i \to [-d; d]$ by

$$f([x]) = \lim_U f_i(x_i).$$

We say that f is the *ultralimit* of the functions $\{f_i\}_{i \in \mathbb{N}}$ and that f is an *ultralimit function.*

Let $\phi(x)$ be a first order formula with a single free variable, and let $f_i^\phi : G_i \to \{0, 1\}$ be defined by

$$f_i^\phi(x) = \begin{cases} 1 & \text{if } G_i \models \phi(x); \\ 0 & \text{otherwise.} \end{cases}$$

and let $f^\phi : \prod_U G_i \to \{0, 1\}$ be defined similarly on the graph $\prod_U G_i$. Then f^ϕ is the ultralimit of the functions $\{f_i^\phi\}$ according to Theorem 16.

The following lemma is proved in [24].

Lemma 13. *The ultralimit functions are measurable on $\prod_U G_i$ and*

$$\int_{\prod_U G_i} f \,\mathrm{d}\nu = \lim_U \frac{\sum_{x \in G_i} f_i(x)}{|G_i|}.$$

In particular, for every formula $\phi(x)$ with a single free variable, we have:

$$\nu\big(\{[x] : \prod_U G_i \models \phi([x])\}\big) = \lim_U \langle \phi, G_i \rangle.$$

Let $\psi(x, y)$ be a formula with two free variables. Define $f_i : G_i \to [0; 1]$ by

$$f_i(x) = \frac{|\{y \in G_i :\ G_i \models \psi(x, y)\}|}{|G_i|}.$$

and let

$$f([x]) = \mu\big(\{[y] : \prod_U G_i \models \psi([x], [y]\}\big).$$

Let us check that $f([x])$ is indeed the ultralimit of $f_i(x_i)$. Fix $[x]$. Let $g_i : G_i \to \{0,1\}$ be defined by

$$g_i(y) = \begin{cases} 1 & \text{if } G_i \models \psi(x_i, y) \\ 0 & \text{otherwise.} \end{cases}$$

and let $g : \prod_U G_i \to \{0,1\}$ be defined similarly by

$$g([y]) = \begin{cases} 1 & \text{if } \prod_U G_i \models \psi([x],[y]) \\ 0 & \text{otherwise.} \end{cases}$$

According to Theorem 16 we have

$$\prod_U G_i \models \psi([x],[y]) \qquad \iff \qquad \{i :\ G_i \models \psi(x_i, y_i)\} \in U.$$

It follows that g is the ultralimit of the functions $\{g_i\}_{i\in\mathbb{N}}$. Thus, according to Lemma 13 we have

$$\nu\big(\{[y] : \prod_U G_I \models \psi([x],[y])\}\big) = \lim_U \frac{|\{y \in G_i : G_i \models \psi(x_i, y_i)\}|}{|G_i|},$$

that is:

$$f([x]) = \lim_U f_i(x_i).$$

Hence f is the ultralimit of the functions $\{f_i\}_{i\in\mathbb{N}}$ and, according to Lemma 13, we have

$$\iint 1_\psi([x],[y])\ \mathrm{d}\nu([x])\ \mathrm{d}\nu([y]) = \lim_U \langle \psi, G_i\rangle.$$

This property extends to any number of free variables and we have the following theorem.

Theorem 18. *Let U be a non-principal ultrafilter and let $(G_n)_{n\in\mathbb{N}}$ be a sequence of graphs.*

Then there the vertex set of the ultraproduct $\prod_U G_i$ can be equipped with a structure of (non-separable) measurable space, and there exists a countably additive measure ν on $\prod_U G_i$ such that for every first-order formula $\phi \in \mathrm{FO}_p$ it holds:

$$\int \cdots \int 1_\phi([x_1], \dots, [x_p])\ \mathrm{d}\nu([x_1])\ \dots\ \mathrm{d}\nu([x_p]) = \lim_U \langle \psi, G_i\rangle.$$

7. Relational sample spaces and Modelings (particularly for bounded degree graphs)

For sparse graphs the appropriate notions of limit objects seem to be relational sample spaces and modelings.

7.1. **Relational sample spaces.** Let us recall Definition 3: Let λ be a signature. A λ-*relational sample space* is a λ-structure $\mathbf{A}$, whose domain A is a standard Borel space with the property that every first-order definable subset of A^p is measurable.

For every integer p, and every $\phi \in \mathrm{FO}_p(\lambda)$ we define

$$\Omega_\phi(\mathbf{A}) = \{(v_1, \dots, v_p) \in A^p :\ \mathbf{A} \models \phi(v_1, \dots, v_p)\}.$$

Formally, a λ-*relational sample space* is a λ-structure $\mathbf{A}$, whose domain A is a standard Borel space such that

$$\forall \phi \in \mathrm{FO}_p(\lambda) \quad \Omega_\phi(\mathbf{A}) \in \Sigma_{\mathbf{A}}^p,$$

where $\Sigma_{\mathbf{A}}$ is the Borel σ-algebra of A.

Lemma 14. *Let λ be a signature, let $\mathbf{A}$ be a λ-structure, whose domain A is a standard Borel space with σ-algebra $\Sigma_{\mathbf{A}}$.*

Then the following conditions are equivalent:

(a) *$\mathbf{A}$ is a λ-relational sample space;*
(b) *for every integer $p \geq 0$ and every $\phi \in \mathrm{FO}_p(\lambda)$, it holds $\Omega_\phi(\mathbf{A}) \in \Sigma_{\mathbf{A}}^p$;*
(c) *for every integer $p \geq 1$ and every $\phi \in \mathrm{FO}_p^{\mathrm{local}}(\lambda)$, it holds $\Omega_\phi(\mathbf{A}) \in \Sigma_{\mathbf{A}}^p$;*
(d) *for every integers $p, q \geq 0$, every $\phi \in \mathrm{FO}_{p+q}(\lambda)$, and every $a_1, \dots, a_q \in A^q$ the set*
$$\{(v_1, \dots, v_p) \in A^p : \ \mathbf{A} \models \phi(a_1, \dots, a_q, v_1, \dots, v_p)\}$$
belongs to $\Sigma_{\mathbf{A}}^p$.

Proof. Items (a) and (b) are equivalent by definition. Also we obviously have the implications (d) $\Rightarrow$ (b) $\Rightarrow$ (c). That (c) $\Rightarrow$ (b) is a direct consequence of Gaifman locality theorem, and the implication (b) $\Rightarrow$ (d) is a direct consequence of Fubini's theorem. □ □

Lemma 15. *Let $\mathbf{A}$ be a relational sample space, let $a \in A$, and let $\mathbf{A}_a$ be the connected component of $\mathbf{A}$ containing a.*

Then $\mathbf{A}_a$ has a measurable domain and, equipped with the σ-algebra of the Borel sets of A included in A_a, it is a relational sample space.

Proof. Let $\phi \in \mathrm{FO}_p^{\mathrm{local}}$ and let
$$X = \{(v_1, \dots, v_p) \in A_a^p : \ \mathbf{A}_a \models \phi(v_1, \dots, v_p)\}.$$
As ϕ is local, there is an integer D such that the satisfaction of ϕ only depends on the D-neighborhoods of the free variables.

For every integer $n \in \mathbb{N}$, denote by $B(\mathbf{A}, a, n)$ the substructure of $\mathbf{A}$ induced by all vertices at distance at most n from a. By the locality of ϕ, for every $v_1, \dots, v_p$ at distance at most n from a it holds
$$\mathbf{A}_a \models \phi(v_1, \dots, v_p) \quad \iff \quad B(\mathbf{A}, a, n + D) \models \phi(v_1, \dots, v_p).$$
However, it is easily checked that there is a local first-order formula $\varphi_n \in \mathrm{FO}_{p+1}^{\mathrm{local}}$ such that for every $v_1, \dots, v_p$ it holds
$$B(\mathbf{A}, a, n{+}D) \models \phi(v_1, \dots, v_p) \wedge \bigwedge_{i=1}^{p} \mathrm{dist}(a, v_i) \leq n \quad \iff \quad \mathbf{A} \models \varphi_n(a, v_1, \dots, v_p).$$

By Lemma 14, it follows that the set $X_n = \{(v_1, \dots, v_n) \in A : \ \mathbf{A} \models \varphi_n(a, v_1, \dots, v_p)\}$ is measurable. As $X = \bigcup_{n \in \mathbb{N}} X_n$, we deduce that X is measurable (with respect to $\Sigma_{\mathbf{A}}^p$. In particular, A_a is a Borel subset of A hence A_a, equipped with the σ-algebra $\Sigma_{\mathbf{A}_a}$ of the Borel sets of A included in A_a, is a standard Borel set. Moreover, it is immediate that a subset of A_a^p belongs to $\Sigma_{\mathbf{A}_a}^p$ if and only if it belongs to $\Sigma_{\mathbf{A}}^p$. Hence, every subset of A_a^p defined by a local formula is measurable with respect to $\Sigma_{\mathbf{A}_a}^p$. By Lemma 14, it follows that $\mathbf{A}_a$ is a relational sample space. □ □

Distinguishing a single element of a λ-relational sample space may be useful in several contexts. This can be achieved, for instance, by adding a new unary symbol to the signature λ, and interpreting the symbol as a marking.

Lemma 16. *Let $\mathbf{A}$ be a λ-relational sample space, let λ^+ be the signature obtained from λ by adding a new unary symbol M and let $\mathbf{A}^+$ be obtained from $\mathbf{A}$ by marking a single $a \in A$ (i.e. a is the only element x of $A^+ = A$ such that $\mathbf{A}^+ \models M(x)$).*

Then $\mathbf{A}^+$ is a relational sample space.

Proof. Let $\phi \in \mathrm{FO}_p(\lambda^+)$. There exists $\phi' \in \mathrm{FO}_{p+1}(\lambda)$ such that for every $x_1, \dots, x_p \in A$ it holds

$$\mathbf{A}^+ \models \phi(x_1, \dots, x_p) \qquad \Longleftrightarrow \qquad \mathbf{A} \models \phi(a, x_1, \dots, x_p).$$

According to Lemma 14, the set of all $(x_1, \dots, x_p)$ such that $\mathbf{A} \models \phi(a, x_1, \dots, x_p)$ is measurable. It follows that $\mathbf{A}^+$ is a relational sample space. □ □

7.2. **Modelings.** Let us recall Definitions 4 and 5: A λ-*modeling* $\mathbf{A}$ is a λ-relational sample space equipped with a probability measure (denoted $\nu_\mathbf{A}$). The *Stone pairing* of $\phi \in \mathrm{FO}(\lambda)$ and a λ-modeling $\mathbf{A}$ is $\langle \phi, \mathbf{A}\rangle = \nu_\mathbf{A}^p(\Omega_\phi(\mathbf{A}))$. Notice that it follows (by Fubini's theorem) that it holds

$$\begin{aligned}\langle \phi, \mathbf{A}\rangle &= \int_{\mathbf{x}\in A^p} 1_{\Omega_\phi(\mathbf{A})}(\mathbf{x})\,\mathrm{d}\nu_\mathbf{A}^p(\mathbf{x})\\ &= \int \cdots \int \mathbf{1}_{\Omega_\phi(\mathbf{A})}(x_1, \dots, x_p)\,\mathrm{d}\nu_\mathbf{A}(x_1)\,\dots\,\mathrm{d}\nu_\mathbf{A}(x_p).\end{aligned}$$

Based on this extension of Stone pairing, we extend our notion of X-convergence.

Definition 9 (modeling X-limit)**.** Let X be a fragment of $\mathrm{FO}(\lambda)$.
If an X-convergent sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of λ-modelings satisfies

$$(\forall \phi \in X) \quad \langle \phi, \mathbf{L}\rangle = \lim_{n\to\infty} \langle \phi, \mathbf{A}_n\rangle$$

for some λ-modeling $\mathbf{L}$, then we say that $\mathbf{L}$ is a *modeling X-limit* of $(\mathbf{A}_n)_{n\in\mathbb{N}}$.

Recall that a λ-modeling $\mathbf{A}$ is *weakly uniform* if all the singletons of A have the same measure. Clearly, every finite λ-structure $\mathbf{A}$ can be identified with the weakly uniform modeling obtained by considering the discrete topology on A. This identification is clearly consistent with our definition of the Stone pairing of a formula and a modeling.

In the case where a modeling $\mathbf{A}$ has an infinite domain, the condition for $\mathbf{A}$ to be weakly uniform is equivalent to the condition for $\nu_\mathbf{A}$ to be atomless. This property is usually fulfilled by modeling X-limits of sequences of finite structures.

Lemma 17. *Let X be a fragment of* FO *that includes* FO_0 *and the formula* $(x_1 = x_2)$*. Then every modeling X-limit of weakly uniform modelings is weakly uniform.*

Proof. Let ϕ be the formula $(x_1 = x_2)$. Notice that for every finite λ-structure $\mathbf{A}$ it holds $\langle \phi, \mathbf{A}\rangle = 1/|A|$ and that for every infinite weakly uniform λ-structure it holds $\langle \phi, \mathbf{A}\rangle = 0$.

Let $\mathbf{L}$ be a modeling X-limit of a sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$. Assume $\lim_{n\to\infty} |A_n| = \infty$. Assume for contradiction that $\nu_\mathbf{L}$ has an atom $\{v\}$ (i.e. $\nu_\mathbf{L}(\{v\}) > 0$). Then $\langle \phi, \mathbf{L}\rangle \geq \nu_\mathbf{L}(\{v\})^2 > 0$, contradicting $\lim_{n\to\infty}\langle \phi, \mathbf{A}_n\rangle = 0$. Hence $\nu_\mathbf{L}$ is atomless.

Otherwise, $|L| = \lim_{n\to\infty} |A_n| < \infty$ (as $\mathbf{L}$ is an elementary limit of $(\mathbf{A}_n)_{n\in\mathbb{N}}$). Let $N = |L|$. Label $v_1, \dots, v_N$ the elements of L and let $p_i = \nu_\mathbf{L}(\{v_i\})$. Then

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N p_i^2 - \Big(\frac{1}{N}\sum_{i=1}^N p_i\Big)^2 &= \frac{\langle \phi, \mathbf{L}\rangle}{N} - \frac{1}{N^2}\\ &= \frac{\lim_{n\to\infty}\langle \phi, \mathbf{A}_n\rangle}{N} - \frac{1}{N^2}\\ &= 0\end{aligned}$$

Thus $p_i = 1/N$ for every $i = 1, \dots, N$. □ □

Corollary 1. *Every modeling* $\mathrm{FO}_2^{\mathrm{local}}$*-limit of finite structures is weakly uniform.*

7.3. **Interpretation Schemes.** In model theory, the notion of *interpretation* of a λ-structure in a κ-structure relies on representing the λ-structure inside the κ-structure: an interpretation of $\mathbf{B}$ in $\mathbf{A}$ is a pair (k, I) where $k \in \mathbb{N}$ and I is a surjective map from a subset of $\mathbf{A}^k$ onto $\mathbf{B}$ such that the preimage by I^k of every first-order definable subset X of B^p is a first-order definable subset of A^{pk} (see e.g. [36, 41]). We shall be interested here in classes of interpretations defined by a common set formulas. This can be formalized as follows.

Definition 10 (Interpretation Scheme)**.** Let κ, λ be signatures, where λ has q relational symbols $R_1, \dots, R_q$ with respective arities $r_1, \dots, r_q$.

An *interpretation scheme* I of λ-structures in κ-structures is defined by an integer k, a formula $E \in \mathrm{FO}_{2k}(\kappa)$, a formula $\theta_0 \in \mathrm{FO}_k(\kappa)$, and a formula $\theta_i \in \mathrm{FO}_{r_i k}(\kappa)$ for each symbol $R_i \in \lambda$, such that:

- the formula E defines an equivalence relation of k-tuples;
- each formula θ_i is compatible with E, in the sense that for every $0 \leq i \leq q$ it holds
$$\bigwedge_{1 \leq j \leq r_i} E(\mathbf{x}_j, \mathbf{y}_j) \quad \vdash \quad \theta_i(\mathbf{x}_1, \dots, \mathbf{x}_{r_i}) \leftrightarrow \theta_i(\mathbf{y}_1, \dots, \mathbf{y}_{r_i}),$$
where $r_0 = 1$, boldface $\mathbf{x}_j$ and $\mathbf{y}_j$ represent k-tuples of free variables, and where $\theta_i(\mathbf{x}_1, \dots, \mathbf{x}_{r_i})$ stands for $\theta_i(x_{1,1}, \dots, x_{1,k}, \dots, x_{r_i,1}, \dots, x_{r_i,k})$.

For a κ-structure $\mathbf{A}$, we denote by $\mathsf{I}(\mathbf{A})$ the λ-structure $\mathbf{B}$ defined as follows:

- the domain B of $\mathbf{B}$ is the subset of the E-equivalence classes $[\mathbf{x}] \subseteq A^k$ of the tuples $\mathbf{x} = (x_1, \dots, x_k)$ such that $\mathbf{A} \models \theta_0(\mathbf{x})$;
- for each $1 \leq i \leq q$ and every $\mathbf{v}_1, \dots, \mathbf{v}_{s_i} \in A^{kr_i}$ such that $\mathbf{A} \models \theta_0(\mathbf{v}_j)$ (for every $1 \leq j \leq r_i$) it holds
$$\mathbf{B} \models R_i([\mathbf{v}_1], \dots, [\mathbf{v}_{r_i}]) \quad \iff \quad \mathbf{A} \models \theta_i(\mathbf{v}_1, \dots, \mathbf{v}_{r_i}).$$

From the standard properties of model theoretical interpretations (see, for instance [41] p. 180), we state the following: if I is an interpretation of λ-structures in κ-structures, then there exists a mapping $\tilde{\mathsf{I}} : \mathrm{FO}(\lambda) \to \mathrm{FO}(\kappa)$ (defined by means of the formulas $E, \theta_0, \dots, \theta_q$ above) such that for every $\phi \in \mathrm{FO}_p(\lambda)$, and every κ-structure $\mathbf{A}$, the following property holds (while letting $\mathbf{B} = \mathsf{I}(\mathbf{A})$ and identifying elements of B with the corresponding equivalence classes of A^k):

For every $[\mathbf{v}_1], \dots, [\mathbf{v}_p] \in B^p$ (where $\mathbf{v}_i = (v_{i,1}, \dots, v_{i,k}) \in A^k$) it holds
$$\mathbf{B} \models \phi([\mathbf{v}_1], \dots, [\mathbf{v}_p]) \quad \iff \quad \mathbf{A} \models \tilde{\mathsf{I}}(\phi)(\mathbf{v}_1, \dots, \mathbf{v}_p).$$

It directly follows from the existence of the mapping $\tilde{\mathsf{I}}$ that an interpretation scheme I of λ-structures in κ-structures defines a continuous mapping from $S(\mathcal{B}(\mathrm{FO}(\kappa)))$ to $S(\mathcal{B}(\mathrm{FO}(\lambda)))$. Thus, interpretation schemes have the following general property:

Proposition 2. *Let I be an interpretation scheme of λ-structures in κ-structures. Then, if a sequence $(\mathbf{A}_n)_{n \in \mathbb{N}}$ of finite κ-structures is* FO*-convergent then the sequence $(\mathsf{I}(\mathbf{A}_n))_{n \in \mathbb{N}}$ of (finite) λ-structures is* FO*-convergent.*

When handling relational sample spaces and modelings, we shall be interested in very simple interpretation schemes, corresponding to the case where $k = 1$, E is equality, and θ_0 is the true statement. In such a context, we can give a simplified definition.

Definition 11. Let κ, λ be signatures. A *basic interpretation scheme* I of λ-structures in κ-structures is defined by a formula $\theta_i \in \mathrm{FO}_{r_i}(\kappa)$ for each symbol $R_i \in \lambda$ with arity r_i.

For a κ-structure $\mathbf{A}$, we denote by $\mathsf{I}(\mathbf{A})$ the structure with domain A such that, for every $R_i \in \lambda$ with arity r_i and every $v_1, \dots, v_{r_i} \in A$ it holds

$$\mathsf{I}(\mathbf{A}) \models R_i(v_1, \dots, v_{r_i}) \quad \iff \quad \mathbf{A} \models \theta_i(v_1, \dots, v_{r_i}).$$

It is immediate that every basic interpretation scheme I defines a mapping $\tilde{\mathsf{I}} : \mathrm{FO}(\lambda) \to \mathrm{FO}(\kappa)$ such that for every κ-structure $\mathbf{A}$, every $\phi \in \mathrm{FO}_p(\lambda)$, and every $v_1, \dots, v_p \in A$ it holds

$$\mathsf{I}(\mathbf{A}) \models \phi(v_1, \dots, v_p) \quad \iff \quad \mathbf{A} \models \tilde{\mathsf{I}}(\phi)(v_1, \dots, v_p).$$

Lemma 18. *A basic interpretation scheme I of λ-structures in κ-structures maps κ-relational sample spaces to λ-relational sample spaces, κ-modelings to λ-modelings, and for every κ-modeling $\mathbf{A}$ and every $\phi \in \mathrm{FO}(\lambda)$ it holds*

$$\langle \phi, \mathsf{I}(\mathbf{A}) \rangle = \langle \tilde{\mathsf{I}}(\phi), \mathbf{A} \rangle.$$

Proof. Assume $\mathbf{A}$ is a κ-relational sample space and $p \in \mathbb{N}$. Every subset of A^p that is first-order definable in $\mathsf{I}(\mathbf{A})$ is first-order definable in $\mathbf{A}$, hence measurable. Thus $\mathsf{I}(\mathbf{A})$ is a λ-relational sample space.

Assume $\mathbf{A}$ is a κ-modeling. Then $\nu_{\mathbf{A}}$ is a probability measure thus so is $\nu_{\mathsf{I}(\mathbf{A})} = \nu_{\mathbf{A}}$. Hence $\mathsf{I}(\mathbf{A})$ is a λ-modeling.

For every $\phi \in \mathrm{FO}_p(\lambda)$ and every $v_1, \dots, v_p \in A$ it holds

$$\mathbf{A} \models \tilde{\mathsf{I}}(\phi)(v_1, \dots, v_p) \quad \iff \quad \mathsf{I}(\mathbf{A}) \models \phi(v_1, \dots, v_p),$$

thus $\langle \tilde{\mathsf{I}}(\phi), \mathbf{A} \rangle = \langle \phi, \mathsf{I}(\mathbf{A}) \rangle$. □ □

The following strengthening of Proposition 2 in the case where we consider a basic interpretation scheme is a clear consequence of Lemma 18.

Proposition 3. *Let I be a basic interpretation scheme of λ-structures in κ-structures.*

If $\mathbf{L}$ is a modeling FO-limit of a sequence $(\mathbf{A}_n)_{n \in \mathbb{N}}$ of κ-modelings then $\mathsf{I}(\mathbf{L})$ is a modeling FO-limit of the sequence $(\mathsf{I}(\mathbf{A}_n))_{n \in \mathbb{N}}$.

Lemma 19. *Let $p \in \mathbb{N}$ be a positive integer, let $\mathbf{L}$ be a modeling, and let ${}^{\mathsf{p}}_{\mathbf{L}}\mathsf{Tp} : L^p \to S(\mathcal{B}(\mathrm{FO}_p(\lambda)))$ be the function mapping $(v_1, \dots, v_p) \in L^p$ to the complete theory of $(\mathbf{L}, v_1, \dots, v_p)$ (that is the set of the formulas $\varphi \in \mathrm{FO}_p(\lambda)$ such that $\mathbf{L} \models \varphi(v_1, \dots, v_p)$).*

Then ${}^{\mathsf{p}}_{\mathbf{L}}\mathsf{Tp}$ is a measurable map from $(L^p, \Sigma^p_{\mathbf{L}})$ to $S(\mathcal{B}(\mathrm{FO}_p(\lambda)))$ (with its Borel σ-algebra).

Let $(\mathbf{A}_n)_{n \in \mathbb{N}}$ be an $\mathrm{FO}_p(\lambda)$-convergent sequence of finite λ-structures, and let μ_p be the associated limit measure (as in Theorem 6).

Then $\mathbf{L}$ is an $\mathrm{FO}_p(\lambda)$-limit modeling of $(\mathbf{A}_n)_{n \in \mathbb{N}}$ if and only if μ_p is the push-forward of the product measure $\nu^p_{\mathbf{L}}$ by the measurable map ${}^{\mathsf{p}}_{\mathbf{L}}\mathsf{Tp}$, that is:

$${}^{\mathsf{p}}_{\mathbf{L}}\mathsf{Tp}_*(\nu^p_{\mathbf{L}}) = \mu_p.$$

Proof. Recall that the clopen sets of $S(\mathcal{B}(\mathrm{FO}_p(\lambda)))$ are of the form $K(\phi)$ for $\phi \in \mathrm{FO}_p(\lambda)$ and that they generate the topology of $S(\mathcal{B}(\mathrm{FO}_p(\lambda)))$ hence also its Borel σ-algebra.

That ${}^{\mathsf{p}}_{\mathbf{L}}\mathsf{Tp}$ is measurable follows from the fact that for every $\phi \in \mathrm{FO}_p$ the preimage of $K(\phi)$, that is ${}^{\mathsf{p}}_{\mathbf{L}}\mathsf{Tp}^{-1}(K(\phi)) = \Omega_\phi(\mathbf{L})$, is measurable.

Assume that $\mathbf{L}$ is an $\mathrm{FO}_p(\lambda)$-limit modeling of $(\mathbf{A}_n)_{n\in\mathbb{N}}$. In order to prove that ${}^{\mathsf{p}}_{\mathbf{L}}\mathsf{Tp}_*(\nu^p_{\mathbf{L}}) = \mu_p$, it is sufficient to check it on sets $K(\phi)$:

$$\mu_p(K(\phi)) = \lim_{n\to\infty}\langle\phi, \mathbf{A}_n\rangle = \langle\phi, \mathbf{L}\rangle = \nu^p_{\mathbf{L}}({}^{\mathsf{p}}_{\mathbf{L}}\mathsf{Tp}^{-1}(K(\phi))).$$

Conversely, if ${}^{\mathsf{p}}_{\mathbf{L}}\mathsf{Tp}_*(\nu^p_{\mathbf{L}}) = \mu_p$ then for every $\phi \in \mathrm{FO}_p(\lambda)$ it holds

$$\langle\phi, \mathbf{L}\rangle = \nu^p_{\mathbf{L}}({}^{\mathsf{p}}_{\mathbf{L}}\mathsf{Tp}^{-1}(K(\phi))) = \mu_p(K(\phi)) = \lim_{n\to\infty}\langle\phi, \mathbf{A}_n\rangle,$$

hence $\mathbf{L}$ is an $\mathrm{FO}_p(\lambda)$-limit modeling of $(\mathbf{A}_n)_{n\in\mathbb{N}}$. □ □

If (X, Σ) is a Borel space with a probability measure ν, it is standard to define the product σ-algebra Σ^ω on the infinite product space $X^{\mathbb{N}}$, which is generated by cylinder sets of the form

$$R = \{f \in L^{\mathbb{N}} :\ f(i_1) \in A_{i_1}, \dots, f(i_k) \in A_{i_k}\}$$

for some $k \in \mathbb{N}$ and $A_{i_1}, \dots, A_{i_k} \in \Sigma$. The measure ν^ω of the cylinder R defined above is then

$$\nu^\omega(R) = \prod_{j=1}^{k} \nu(A_{i_j}).$$

By Kolmogorov's Extension Theorem, this extends to a unique probability measure on Σ^ω (which we still denote by ν^ω). We summarize this as the following (see also Fig. 7.3).

Theorem 19. *let* $\mathbf{L}$ *be a modeling, and let* ${}^{\omega}_{\mathbf{L}}\mathsf{Tp} : L^{\mathbb{N}} \to S(\mathcal{B}(\mathrm{FO}(\lambda)))$ *be the function mapping* $f \in L^{\mathbb{N}}$ *to the point of* $S(\mathcal{B}(\mathrm{FO}(\lambda)))$ *corresponding to the set* $\{\phi : \mathbf{L} \models \phi(f(1), \dots, f(i), \dots)\}$.

Then ${}^{\omega}_{\mathbf{L}}\mathsf{Tp}$ *is a measurable map.*

Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *be an* $\mathrm{FO}(\lambda)$*-convergent sequence of finite* λ*-structures, and let* μ *be the associated limit measure (see Theorem 5).*

Then $\mathbf{L}$ *is an* $\mathrm{FO}(\lambda)$*-limit modeling of* $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *if and only if*

$${}^{\omega}_{\mathbf{L}}\mathsf{Tp}_*(\nu^{\omega}_{\mathbf{L}}) = \mu.$$

□

$$\begin{tikzcd}
 & \begin{array}{c}L\\ \text{with } \nu_{\mathbf{L}}\end{array} \arrow[d, "{}^{1}_{\mathbf{L}}\mathsf{Tp}"'] & \begin{array}{c}L^2\\ \text{with } \nu^2_{\mathbf{L}}\end{array} \arrow[l] \arrow[loop above, "S_2"] \arrow[d, "{}^{2}_{\mathbf{L}}\mathsf{Tp}"'] & \dots \arrow[l] & \begin{array}{c}L^p\\ \text{with } \nu^p_{\mathbf{L}}\end{array} \arrow[l] \arrow[loop above, "S_p"] \arrow[d, "{}^{\mathsf{p}}_{\mathbf{L}}\mathsf{Tp}"'] & \begin{array}{c}L^{\mathbb{N}}\\ \text{with } \nu^\omega_{\mathbf{L}}\end{array} \arrow[l] \arrow[loop above, "S_\omega"] \arrow[d, "{}^{\omega}_{\mathbf{L}}\mathsf{Tp}"'] \\
\begin{array}{c}S(\mathcal{B}(\mathrm{FO}_0))\\ \text{with } \delta_T\end{array} & \begin{array}{c}S(\mathcal{B}(\mathrm{FO}_1))\\ \text{with } \mu_1\end{array} \arrow[l] & \begin{array}{c}S(\mathcal{B}(\mathrm{FO}_2))\\ \text{with } \mu_2\end{array} \arrow[l] \arrow[loop above, "S_2"] \arrow[d] & \dots \arrow[l] & \begin{array}{c}S(\mathcal{B}(\mathrm{FO}_p))\\ \text{with } \mu_p\end{array} \arrow[l] \arrow[loop above, "S_p"] \arrow[d] & \begin{array}{c}S(\mathcal{B}(\mathrm{FO}))\\ \text{with } \mu\end{array} \arrow[l] \arrow[loop above, "S_\omega"] \arrow[d] \\
 & & \begin{array}{c}S(\mathcal{B}(\mathrm{FO}_1))^2\\ \text{with } \mu^2_1\end{array} \arrow[ul] \arrow[loop above, "S_2"] & \dots \arrow[l] & \begin{array}{c}S(\mathcal{B}(\mathrm{FO}_1))^p\\ \text{with } \mu^p_1\end{array} \arrow[l] \arrow[loop above, "S_p"] & \begin{array}{c}S(\mathcal{B}(\mathrm{FO}_1))^{\mathbb{N}}\\ \text{with } \mu^\omega_1\end{array} \arrow[l] \arrow[loop above, "S_\omega"]
\end{tikzcd}$$

FIGURE 3. Pushforward of measures

Remark 5. We could have considered that free variables are indexed by $\mathbb{Z}$ instead of $\mathbb{N}$. In such a context, natural shift operations S and T act respectively on the Stone space $\mathcal{S}$ of the Lindendaum-Tarski algebra of $\mathrm{FO}(\lambda)$, and on the space $\mathbf{L}^{\mathbb{Z}}$ of the mappings from $\mathbb{Z}$ to a λ-modeling $\mathbf{L}$. If $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is an FO-convergent sequence with limit measure μ on $\mathcal{S}$, then $(\mathcal{S},\mu,S)$ is a measure-preserving dynamical system. Also, if $\nu^{\mathbb{Z}}$ is the product measure on $\mathbf{A}$, $(\mathbf{A}^{\mathbb{Z}},\nu,T)$ is a Bernoulli scheme. Then, the condition of Theorem 19 can be restated as follows: the modeling $\mathbf{L}$ is a modeling FO-limit of the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ if and only if $(\mathcal{S},\mu,S)$ is a factor of $(\mathbf{A}^{\mathbb{Z}},\nu^{\mathbb{Z}},T)$. This setting leads to yet another interpretation of our result, which we hope will be treated elsewhere.

7.4. **Modeling FO-limits for Graphs of Bounded Degrees.** Nice limit objects are known for sequence of bounded degree connected graphs, both for BS-convergence (graphing) and for FO_0-convergence (countable graphs). It is natural to ask whether a nice limit object could exist for full FO-convergence. We shall now answer this question by the positive. First we take time to comment on the connectivity assumption. A first impression is that FO-convergence of disconnected graphs could be considered component-wise. The following examples shows that this is far from being true in general. The contrast between the behaviour of graphs with a first-order definable component relation (like graphs with bounded diameter components) and of graphs with bounded degree is exemplified by the following example.

Example 4. Consider a BS-convergent sequence $(G_n)_{n\in\mathbb{N}}$ of planar graphs with bounded degrees such that the limit distribution has an infinite support. Note that $\lim_{n\to\infty}|G_n|=\infty$. Then, as planar graphs with bounded degrees form a hyperfinite class of graphs there exists, for every graph G_n and every $\epsilon>0$ a subgraph $S(G_n,\epsilon)$ of G_n obtained by deleting at most $\epsilon|G_n|$ of edges, such that the connected components of $S(G_n,\epsilon)$ have order at most $f(\epsilon)$. By considering a subsequence $G_{s(n)}$ we can assume $\lim_{n\to\infty}|G_{s(n)}|/f(1/n)=\infty$. Then note that the sequences $(G_s(n))_{n\in\mathbb{N}}$ and $(S(G_{s(n)},1/n))_{n\in\mathbb{N}}$ have the same BS-limit. By merging these sequences, we conclude that there exists an $\mathrm{FO}^{\mathrm{local}}$ convergent sequence of graphs with bounded degrees (H_n) such that H_n is connected if n is even and such that the number of connected components of H_n for n odd tends to infinity.

Example 5. Consider four sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(C_n)_{n\in\mathbb{N}}$,$(D_n)_{n\in\mathbb{N}}$ of FO-converging sequences where $|A_n|=|B_n|=|C_n|=|D_n|$ grows to infinity, the sequences have distinct limits, and each of A_n,B_n,C_n,D_n contains an induced path of length n. Then we can construct a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with two connected components $H_{n,1}$ and $H_{n,2}$ obtained by cutting the induced paths in A_n,B_n,C_n and D_n in their middle and alternatively gluing A_n with C_n and B_n with D_n, or A_n with D_n and B_n with C_n (see Fig. 4). Then $(G_n)_{n\in\mathbb{N}}$ is FO-convergent. However, there is no choice of a mapping $f:\mathbb{N}\to\{1,2\}$ such that $(H_{n,f(n)})$ is FO-convergent (or even BS-convergent).

This situation is indeed related to the fact that the diameter of the graph G_n in the sequence tend to infinity as n grows and that the belonging to a same connected component cannot be defined by a first-order formula. This situation is standard when one consider BS-limits of connected graphs with bounded degrees: it is easily checked that, as a limit of connected graphs, a graphing may have uncountably many connected components.

Let V be a standard Borel space with a measure μ. Suppose that $T_1,T_2,\dots,T_k$ are measure preserving Borel involutions of X. Then the system

$$\mathbf{G}=(V,T_1,T_2,\dots,T_k,\mu)$$

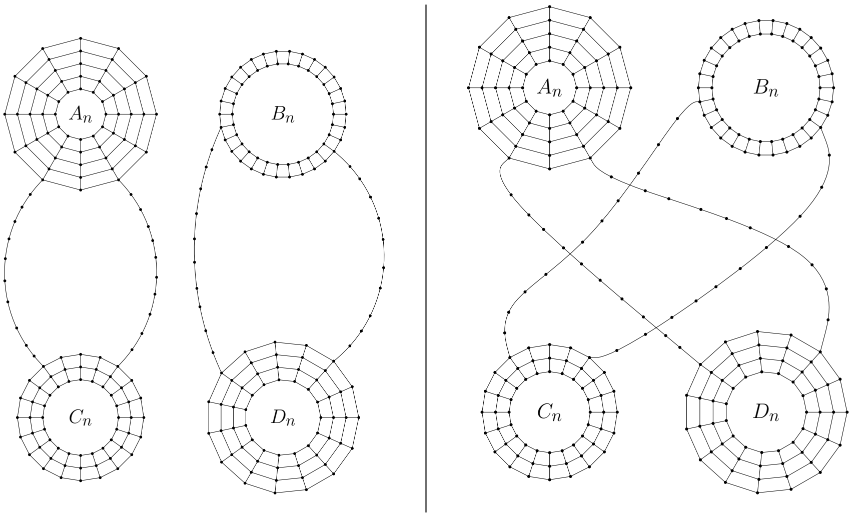

FIGURE 4. An FO-converging sequence with no component selection

is called a *measurable graphing* (or simply a *graphing*) [1]. A graphing $\mathbf{G}$ determines an equivalence relation on the points of V. Simply, $x \sim_{\mathbf{G}} y$ if there exists a sequence of points $(x_1, x_2, \dots, x_m)$ of X such that

- $x_1 = x, x_m = y$
- $x_{i+1} = T_j(x_i)$ for some $1 \leq j \leq k$.

Thus there exist natural a simple graph structure on the equivalence classes, the *leafgraph*. Here x is adjacent to y, if $x \neq y$ and $T_j(x) = y$ for some $1 \leq j \leq k$. Now if If V is a compact metric space with a Borel measure μ and $T_1, T_2, \dots, T_k$ are continuous measure preserving involutions of V, then $\mathbf{G} = (V, T_1, T_2, \dots, T_k, \mu)$ is a *topological graphing*. It is a consequence of [6] and [29] that every local weak limit of finite connected graphs with maximum degree at most D can be represented as a measurable graphing. Elek [23] further proved the representation can be required to be a topological graphing.

A graphing defines an edge coloration, where $\{x, y\}$ is colored by the set of the indexes i such that $y = T_i(x)$. For an integer r, a graphing $\mathbf{G} = (V, T_1, \dots, T_k, \mu)$ and a finite rooted edge colored graph (F, o) we define the set

$$D_r(\mathbf{G}, (F, o)) = \{x \in \mathbf{G}, B_r(\mathbf{G}, x) \simeq (F, o)\}.$$

It is easily checked that $D_r(\mathbf{G}, (F, o))$ is measurable.

Considering k-edge colored graphing allows to describe a vertex x in a distance-r neighborhood of a given vertex v by the sequence of the colors of the edges of a path linking v to x. Taking, among the minimal length sequences, the one which is lexicographically minimum, it is immediate that for every vertex v and every integer r there is a injection $\iota_{v,r}$ from $B_r(\mathbf{G}, v)$ to the set of the sequences of length at most r with values in $[k]$. Moreover, if $B_r(\mathbf{G}, v)$ and $B_r(\mathbf{G}, v')$ are isomorphic as edge-colored rooted graphs, then there exists a unique isomorphism $f : B_r(\mathbf{G}, v) \to B_r(\mathbf{G}, v')$ and this isomorphism as the property that for every $x \in B_r(\mathbf{G}, v)$ it holds $\iota_{v,r'}(f(x)) = \iota_{v,r}(x)$.

Lemma 20. *Every graphing is a modeling.*

Proof. Let $\mathbf{G} = (V, T_1, \dots, T_d, \mu)$ be a graphing. We color the edges of G according to the the involutions involved.

For $r \in \mathbb{N}$, we denote by $\mathcal{F}_r$ the finite set of all the colored rooted graphs that arise as $B_r(\mathbf{G}, v)$ for some $v \in V$. To every vertex $v \in V$ and integer $r \in \mathbb{N}$ we associate $t_r(v)$, which is the isomorphism type of the edge colored ball $B_r(\mathbf{G}, v)$.

According to Gaifman's locality theorem, in order to prove that $\mathbf{G}$ is a modeling, it is sufficient to prove that for each $\phi \in FO_p^{\mathrm{local}}$, the set

$$X = \{(v_1, \dots, v_p) \in V^q : \quad \mathbf{G} \models \phi(v_1, \dots, v_p)\}$$

is measurable (with respect to the product σ-algebra of V^p).

Let $L \in \mathbb{N}$ be such that ϕ is L-local. For every $\mathbf{v} = (v_1, \dots, v_p) \in X$ we define the graph $\Gamma(\mathbf{v})$ with vertex set $\{v_1, \dots, v_p\}$ such that two vertices of $\Gamma(\mathbf{v})$ are adjacent if their distance in $\mathbf{G}$ is at most L. We define a partition $\mathcal{P}(\mathbf{v})$ of $[p]$ as follows: i and j are in a same part if v_i and v_j belong to a same connected component of $\Gamma(v)$. To each part $P \in \mathcal{P}(\mathbf{v})$, we associate the tuple formed by $T_P = t_{(|P|-1)L}(v_{\min P})$ and, for each $i \in P - \{\min P\}$, a composition $F_{P,i} = T_{i_1} \circ \dots \circ T_{i_j}$ with $1 \le j \le (|P|-1)L$, such that $v_i = F_{P,i}(v_{\min P})$. We also define $F_{P,\min P}$ as the identity mapping. According to the locality of ϕ, if $\mathbf{v}' = (v_1', \dots, v_p') \in V^p$ defines the same partition, types, and compositions, then $\mathbf{v}' \in X$. For fixed partition $\mathcal{P}$, types $(T_P)_{P \in \mathcal{P}}$, and compositions $(F_{P,i})_{i \in P \in \mathcal{P}}$, the corresponding subset X' of X is included in a (reshuffled) product Y of sets of tuples of the form $(F_{P,i}(x_{\min P}))$ for $v_{\min P} \in W_P$, and is the set of all $v \in G$ such that $B_{(|P|-1)L}(\mathbf{G}, v) = T_P$. Hence W_P is measurable and (as each $F_{P,i}$ is measurable) Y is a measurable subset of $G^{|P|}$. Of course, this product may contain tuples $\mathbf{v}$ defining another partition. A simple induction and inclusion/exclusion argument shows that X' is measurable. As X is the union of a finite number of such sets, X is measurable. □ □

We shall make use of the following lemma which reduces a graphing to its essential support.

Lemma 21 (Cleaning Lemma)**.** *Let $\mathbf{G} = (V, T_1, \dots, T_d, \mu)$ be a graphing.*

Then there exists a subset $X \subset V$ with 0 measure such that X is globally invariant by each of the T_i and $\mathbf{G}' = (V - X, T_1, \dots, T_d, \mu)$ is a graphing such that for every finite rooted colored graph (F, o) and integer r it holds

$$\mu(D_r(\mathbf{G}', (F, o))) = \mu(D_r(\mathbf{G}, (F, o)))$$

(which means that $\mathbf{G}'$ is equivalent to $\mathbf{G}$) and

$$D_r(\mathbf{G}', (F, o)) \neq \emptyset \quad \iff \quad \mu(D_r(\mathbf{G}', (F, o))) > 0.$$

Proof. For a fixed r, define $\mathcal{F}_r$ has the set of all (isomorphism types of) finite rooted k-edge colored graphs (F, o) with radius at most r such that $\mu(D_r(\mathbf{G}, (F, o))) = 0$. Define

$$X = \bigcup_{r \in \mathbb{N}} \bigcup_{(F,o) \in \mathcal{F}_r} D_r(\mathbf{G}, (F, o)).$$

Then $\mu(X) = 0$, as it is a countable union of 0-measure sets.

We shall now prove that X is a union of connected components of $\mathbf{G}$, that is that X is globally invariant by each of the T_i. Namely, if $x \in X$ and y is adjacent to x, then $y \in X$. Indeed: if $x \in X$ then there exists an integer r such that $\mu(D(\mathbf{G}, B_r(\mathbf{G}, x))) = 0$. But it is easily checked that

$$\mu(D(\mathbf{G}, B_{r+1}(\mathbf{G}, y))) \le d \cdot \mu(D(\mathbf{G}, B_r(\mathbf{G}, x))).$$

Hence $y \in X$. It follows that for every $1 \le i \le d$ we have $T_i(X) = X$. So we can define the graphing $\mathbf{G}' = (V - X, T_1, \dots, T_d, \mu)$.

Let (F, o) be a rooted finite colored graph. Assume there exists $x \in \mathbf{G}'$ such that $B_r(\mathbf{G}', r) \simeq (F, o)$. As X is a union of connected components, we also have $B_r(\mathbf{G}, r) \simeq (F, o)$ and $x \notin X$. It follows that $\mu(D(\mathbf{G}, (F, o))) > 0$ hence it holds $\mu(D_r(\mathbf{G}', (F, o))) > 0$. □ □

The cleaning lemma allows us a clean description of FO-limits in the bounded degree case:

Theorem 20. *Let $(G_n)_{n\in\mathbb{N}}$ be a FO-convergent sequence of finite graphs with maximum degree d, with $\lim_{n\to\infty} |G_n| = \infty$. Then there exists a graphing $\mathbf{G}$, which is the disjoint union of a graphing $\mathbf{G}_0$ and a countable graph $\hat{G}$ such that*

- *The graphing $\mathbf{G}$ is a modeling* FO*-limit of the sequence $(G_n)_{n\in\mathbb{N}}$.*
- *The graphing $\mathbf{G}_0$ is a BS-limit of the sequence $(G_n)_{n\in\mathbb{N}}$ such that*

$$D_r(\mathbf{G}_0, (F, o)) \neq \emptyset \iff \mu(D_r(\mathbf{G}_0, (F, o))) > 0.$$

- *The countable graph $\hat{G}$ is an elementary limit of the sequence $(G_n)_{n\in\mathbb{N}}$.*

Proof. Let $\mathbf{G}_0$ be a BS-limit, which has been "cleaned" using the previous lemma, and let $\hat{G}$ be an elementary limit of G. It is clear that $\mathbf{G} = \mathbf{G}_0 \cup \hat{G}$ is also a BS-limit of the sequence, so the lemma amounts in proving that $\mathbf{G}$ is elementarily equivalent to $\hat{G}$.

According to Hanf's theorem [33], it is sufficient to prove that for every integers r, t and for every rooted finite graph (F, o) (with maximum degree d) the following equality holds:

$$\min(t, |D_r(\mathbf{G}, (F, o))|) = \min(t, |D_r(\hat{G}, (F, o))|).$$

Assume for contradiction that this is not the case. Then $|D_r(\hat{G}, (F, o))| < t$ and $D_r(\mathbf{G}_0, (F, o))$ is not empty. However, as $\mathbf{G}_0$ is clean, this implies $\mu(D_r(\mathbf{G}_0, (F, o))) = \alpha > 0$. It follows that for every sufficiently large n it holds $|D_r(G_n, (F, o))| > \alpha/2\,|G_n| > t$. Hence $|D_r(\hat{G}, (F, o))| > t$, contradicting our hypothesis.

That $\mathbf{G}$ is a modeling then follows from Lemma 20. □ □

Remark 6. Not every graphing with maximum degree 2 is an FO-limit modeling of a sequence of finite graphs. Indeed: let $\mathbf{G}$ be a graphing that is an FO-limit modeling of the sequence of cycles. The disjoint union of $\mathbf{G}$ and a ray is a graphing $\mathbf{G}'$, which has the property that all its vertices but one have degree 2, the exceptional vertex having degree 1. As this property is not satisfied by any finite graph, $\mathbf{G}'$ is not the FO-limit of a sequence of finite graphs.

Let us finish this section by giving an interesting example, which shows that the cleaning lemma sometimes applies in a non-trivial way:

Example 6. Consider the graph G_n obtained from a De Bruijn sequence of length 2^n as shown Fig 5.

It is easy to define a graphing $\mathbf{G}$, which is the limit of the sequence $(G_n)_{n\in\mathbb{N}}$: as vertex set, we consider the rectangle $[0;1) \times [0;3)$. We define a measure preserving

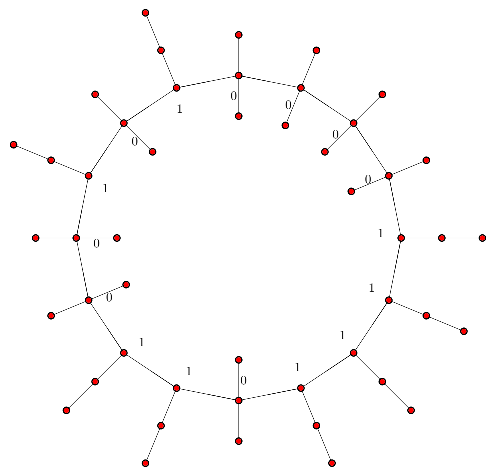

FIGURE 5. The graph G_n is constructed from a De Bruijn sequence of length 2^n.

function f and two measure preserving involutions T_1, T_2 as follows:

$$f(x,y) = \begin{cases} (2x, y/2) & \text{if } x < 1/2 \text{ and } y < 1 \\ (2x-1, (y+1)/2) & \text{if } 1/2 \leq x \text{ and } y < 1 \\ (x,y) & \text{otherwise} \end{cases}$$

$$T_1(x,y) = \begin{cases} (x, y+1) & \text{if } y < 1 \\ (x, y-1) & \text{if } 1 \leq y < 2 \\ (x,y) & \text{otherwise} \end{cases}$$

$$T_2(x,y) = \begin{cases} (x, y+1) & \text{if } x < 1/2 \text{ and } 1 \leq y < 2 \\ (x, y+2) & \text{if } 1/2 \leq x \text{ and } y < 1 \\ (x, y-1) & \text{if } x < 1/2 \text{ and } 2 \leq y \\ (x, y-2) & \text{if } 1/2 \leq x \text{ and } 2 \leq y \\ (x,y) & \text{otherwise} \end{cases}$$

Then the edges of $\mathbf{G}$ are the pairs $\{(x,y),(x',y')\}$ such that $(x,y) \neq (x',y')$ and either $(x',y') = f(x,y)$, or $(x,y) = f(x',y')$, or $(x',y') = T_1(x,y)$, or $(x',y') = T_2(x,y)$.

If one considers a random root (x,y) in $\mathbf{G}$, then the connected component of (x,y) will almost surely be a rooted line with some decoration, as expected from what is seen from a random root in a sufficiently large G_n. However, special behaviour may happen when x and y are rational. Namely, it is possible that the connected component of (x,y) becomes finite. For instance, if $x = 1/(2^n - 1)$ and $y = 2^{n-1}x$ then the orbit of (x,y) under the action of f has length n thus the connected component of (x,y) in $\mathbf{G}$ has order $3n$. Of course, such finite connected components do not appear in G_n. Hence, in order to clean $\mathbf{G}$, infinitely many components have to be removed.

Let us give a simple example exemplifying the distinction between BS and FO-convergence for graphs with bounded degree.

Example 7. Let G_n denote the $n \times n$ grid. The Benjamini-Schramm limit object is a probability distribution concentrated on the infinite grid with a specified root. A limit graphing can be described as the Lebesgue measure on $[0,1]^2$, where (x, y) is adjacent to $(x \pm \alpha \bmod 1, y \pm \alpha \bmod 1)$ for some irrational number α.

This graphing, however, is not an FO-limit of the sequence $(G_n)_{n\in\mathbb{N}}$ as every FO-limit has to contain four vertices of degree 2. An FO-limit graphing can be described as the above graphing restricted to $[0,1)^2$ (obtained by deleting all vertices with $x = 1$ or $y = 1$). One checks for instance that this graphing contains four vertices of degree 2 (the vertices (α, α), $(1-\alpha, \alpha)$, $(\alpha, 1-\alpha)$, and $(1-\alpha, 1-\alpha)$) and infinitely many vertices of degree 3.

We want to stress that our general and unifying approach to structural limits was not developed for its own sake and that it provided a proper setting (and, yes, encouragement) for the study of classes of sparse graphs. So far the bounded degree graphs are the only sparse class of graphs where the structural limits were constructed efficiently. (Another example of limits of sparse graphs is provided by scaling limits of transitive graphs [5] which proceeds in different direction and is not considered here.) The goal of the remaining sections of this article is to extend this to strong Borel FO-limits of rooted trees with bounded height and thus, by means of a fitting basic interpretation scheme, to graphs with bounded tree-depth (defined in [50]), or graphs with bounded SC-depth (defined in [31]).

8. Merging Limits: Combinations of Modelings

The combinatorics of limits of equivalence relations (such as components) is complicated. As a first approximation towards analysis we consider the combinatorics of "large" equivalence classes. This leads to the notion of spectrum, which will be analyzed in this section.

8.1. Spectrum of a first-order equivalence relation.

Definition 12 (ϖ-spectrum)**.** Let $\mathbf{A}$ be a λ-modeling (with measure $\nu_{\mathbf{A}}$), and let $\varpi \in \mathrm{FO}_2(\lambda)$ be an equivalence relation on A. Let $\{C_i : i \in \Gamma\}$ be set of all the ϖ-equivalence classes of A, and let Γ_+ be the (at most countable) subset of Γ of the indexes i such that $\nu_{\mathbf{A}}(C_i) > 0$.

The *ϖ-spectrum* $\mathrm{Sp}_\varpi(\mathbf{A})$ of $\mathbf{A}$ is the (at most countable) sequence of the values $\nu_{\mathbf{A}}^p(C_i)$ (for $i \in \Gamma_+$) ordered in non-increasing order.

Lemma 22. *For $k \in \mathbb{N}$, let $\varpi^{(k)}$ be the formula $\bigwedge_{i=1}^k \varpi(x_i, x_{i+1})$. Then it holds*

$$\sum_{i\in\Gamma_+} \nu_{\mathbf{A}}(C_i)^{k+1} = \langle \varpi^{(k)}, \mathbf{A}\rangle.$$

Proof. Let $k \in \mathbb{N}$. Define

$$D_{k+1} = \{(x_1, \dots, x_{k+1}) \in A^{k+1} :\ A \models \varpi^k(x_1, \dots, x_{k+1})\}.$$

According to Lemma 15, each C_i is measurable, thus $\bigcup_{i\in\Gamma_+} C_i$ is measurable and so is $R = A \setminus \bigcup_{i\in\Gamma_+} C_i$.

Considering the indicator function $\mathbf{1}_{D_{k+1}\cap R^{k+1}}$ of $D_{k+1} \cap R^{k+1}$ and applying Fubini's theorem, we get

$$\int_{A^{k+1}} \mathbf{1}_{D_{k+1}\cap R^{k+1}}\, \mathrm{d}\nu_{\mathbf{A}}^{k+1} = \int \cdots \int \mathbf{1}_R(x_1, \dots, x_{k+1})\ \mathrm{d}\nu_{\mathbf{A}}(x_1, \dots, \mathrm{d}\nu_{\mathbf{A}}(x_{k+1}) = 0.$$

as for every fixed $a_1, \dots, a_k$ (with $a_1 \in C_\alpha$, for some $\alpha \in \Gamma \setminus \Gamma_+$) we have

$$0 \le \int \mathbf{1}_R(a_1, \dots, a_k, x_{k+1})\, \mathrm{d}\nu_{\mathbf{A}}(x_{k+1}) \le \nu_{\mathbf{A}}(C_\alpha) = 0.$$

It follows (by countable additivity) that

$$\langle \varpi^{(k)}, \mathbf{A}\rangle = \nu_{\mathbf{A}}^{k+1}(D_{k+1}) = \nu_{\mathbf{A}}^{k+1}\Big(\bigcup_{i\in\Gamma_+} C_i^{k+1}\Big) = \sum_{i\in\Gamma_+} \nu_{\mathbf{A}}(C_i)^{k+1}.$$

□ □

It follows from Lemma 22 that the spectrum $\mathrm{Sp}_\varpi(\mathbf{A})$ is computable from the sequence of (non-increasing) values $(\langle \varpi^{(k)}, \mathbf{A}\rangle)_{k\in\mathbb{N}}$.

We assume that every finite sequence $\mathbf{x} = (x_1, \dots, x_n)$ of positive reals is implicitly embedded in an infinite sequence by defining $x_i = 0$ for $i > n$. Recall the usual ℓ_k norms:

$$\|\mathbf{x}\|_k = \Big(\sum_i |x_i|^k\Big)^{1/k}.$$

Hence above equations rewrite as

$$\|\mathrm{Sp}_\varpi(\mathbf{A})\|_{k+1} = \langle \varpi^{(k)}, \mathbf{A}\rangle^{1/(k+1)} \tag{4}$$

We shall prove that the spectrum is, in a certain sense, defined by a continuous function. We need the following technical lemma.

Lemma 23. *For each $n \in \mathbb{N}$, let $\mathbf{a}_n = (a_{n,i})_{i\in\mathbb{N}}$ be a non-increasing sequence of positive real numbers with bounded sum (i.e $\|\mathbf{a}_n\|_1 < \infty$ for every $n \in \mathbb{N}$).*

Assume that for every integer $k \geq 1$ the limit $s_k = \lim_{n\to\infty} \|\mathbf{a}_n\|_k$ exists.

Then $(\mathbf{a}_n)_{n\in\mathbb{N}}$ converges in the space c_0 of all sequences converging to zero (with norm $\|\cdot\|_\infty$).

Proof. We first prove that the sequences converge pointwise, that is that there exists a sequence $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$ it holds

$$x_i = \lim_{n\to\infty} a_{n,i}.$$

For every $\epsilon > 0$, if $s_k < \epsilon$ then $a_{n,1} < 2\epsilon$ for all sufficiently large values of n. Thus if $s_k = 0$ for some k, the limit $\lim_{n\to} a_{n,i}$ exists for every i and is null. Thus, we can assume that s_k is strictly positive for every $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that for every $n \geq N$ it holds $|s_k^k - \|\mathbf{a}_n\|^k| < s_k^k/k$. As $(a_{n,i})_{i\in\mathbb{N}}$ is a non-increasing sequence of positive real numbers, for every $n \neq N$ it holds

$$a_{n,1}^k \leq \|\mathbf{a}_n\|^k < s_k^k(1 + 1/k)$$

and

$$a_{n,1}^{k-1} \geq \|\mathbf{a}_n\|^k > s_k^k(1 - 1/k)$$

Hence

$$\log s_k + \frac{\log(1+1/k)}{k} \geq \log a_{n,1} \geq \big(1 + \frac{1}{k-1}\big)\big(\log s_k + \frac{\log(1-1/k)}{k}\big)$$

Thus $x_1 = \lim_{n\to\infty} a_{n,1}$ exists and $x_1 = \lim_{k\to\infty} s_k$. Inductively, we get that for each $i \in \mathbb{N}$, the limit $x_i = \lim_{n\to\infty} a_{n,i}$ exists and that

$$x_i = \lim_{k\to\infty} (s_k^k - \sum_{j<i} x_j^k)^{1/k}.$$

We now prove that the converge is uniform, that is that for every $\epsilon > 0$ there exists N such that for every $n \geq N$ it holds

$$\|\mathbf{x} - \mathbf{a}_n\|_\infty < \epsilon.$$

As $\mathbf{a}_n \in \ell_1$ and $\|\mathbf{a}_n\|_1$ converges there exists M such that $\|\mathbf{a}_n\|_1 \leq M$ for every $n \in \mathbb{N}$. Let $\epsilon > 0$. Let $A = \min\{i : x_i \leq \epsilon/3\}$. (Note that $A \leq 3M/\epsilon$.) There exists

N such that for every $n \geq N$ it holds $\sup_{i \leq A} |x_i - a_{n,i}| < \epsilon/3$. Moreover, for every $i > A$ it holds

$$0 \leq a_{n,i} \leq a_{n,A} < x_A + \epsilon/3 < 2\epsilon/3.$$

As $0 \leq \lambda_i \leq \epsilon/3$ for every $i > A$ it holds

$$|x_i - a_{n,i}| < \epsilon$$

for every $i > A$ (hence for every i). Thus $(\mathbf{a}_n)_{n\in\mathbb{N}}$ converges in ℓ_∞. As obviously each $\mathbf{a}_n$ has 0 limit, $(\mathbf{a}_n)_{n\in\mathbb{N}}$ converges in c_0. □ □

Lemma 24. *Let λ be a signature. The mapping $\mathbf{A} \mapsto \mathrm{Sp}_\varpi(\mathbf{A})$ is a continuous mapping from the space of λ-modelings with component relation ϖ (with the topology of $\mathrm{FO}^{\mathrm{local}}(\lambda)$-convergence) to the space c_0 of all sequences converging to zero (with $\|\cdot\|_\infty$ norm).*

Proof. Assume $\mathbf{A}_n$ is an $\mathrm{FO}^{\mathrm{local}}(\lambda)$-convergent sequence of λ-modelings.

Let $(\lambda_{n,1}, \dots, \lambda_{n,i}, \dots)$ be the ϖ-spectrum of $\mathbf{A}_n$ (extended by zero values if finite), and let $\mathbf{a}_n = (a_{n,i})_{i\in\mathbb{N}}$ be the sequence defined by $a_{n,i} = \lambda_{n,i}^2$. Then for every integer $k \geq 1$ it holds

$$\|a_n\|_k = \|\mathrm{Sp}_\varpi(\mathbf{A}_n)\|_{2k}^2 = \langle \varpi^{(2k-1)}, \mathbf{A}_n\rangle^{1/k}.$$

Hence $s_k = \lim_{n\to\infty} \|a_n\|_k$ exists. According to Lemma 23, $(\mathbf{a}_n)_{n\in\mathbb{N}}$ converges in c_0, thus so does $(\mathrm{Sp}_\varpi(\mathbf{A}_n))_{n\in\mathbb{N}}$. □ □

8.2. **Component-Local Formulas.**

Definition 13 (Component relation)**.** Let λ be a signature and let $\mathbf{A}$ be a λ-relational structure.

A binary relation $\varpi \in \lambda$ is a *component relation* of $\mathbf{A}$ if it is complete on the connected components of $\mathbf{A}$.

The property of ϖ to be a component relation can be axiomatized by a sentence and we shall denote by $\mathcal{K}_\varpi$ the class of all λ-structures for which ϖ is a component relation. Note that in some applications, the relation ϖ can be defined through a basic interpretation scheme.

A local formula ϕ with p free variables is *ϖ-local* if ϕ is equivalent (in $\mathcal{K}_\varpi$) to $\phi \wedge \bigwedge_{i=1}^{p-1} \varpi(x_i, x_{i+1})$. For two ϖ-local formulas ϕ_1, ϕ_2 we denote by $\phi_1 \mathbin{\triangle_\varpi} \phi_2$ the formula $\phi_1 \wedge \phi_2 \wedge \varpi(x_a, x_b)$ where $x_a \in \mathrm{Fv}(\phi_1)$ and $x_b \in \mathrm{Fv}(\phi_2)$. Notice that $\phi_1 \mathbin{\triangle_\varpi} \phi_2$ is, by construction, a ϖ-local formula.

Recall that $\mathfrak{S}_n$ denotes the symmetric group of $\{1, \dots, n\}$. For a permutation $\sigma \in \mathfrak{S}_n$, we denote by $[n]/\sigma$ the set of the orbits of σ. Elements of $[n]/\sigma$ (that is: orbits) are identified with the corresponding subsets of $[n]$.

The basis observation is that for ϖ-local formulas, we can reduce the Stone pairing to components.

Lemma 25. *Let $\mathbf{A}$ be a λ-modeling and component relation ϖ. Let $\psi \in \mathrm{FO}_p(\lambda)$ be a ϖ-local formula of $\mathbf{A}$.*

Assume $\mathbf{A}$ has countably many connected components $\{\mathbf{A}_i\}_{i\in\Gamma}$. Let Γ_+ be the set of indexes i such that $\nu_{\mathbf{A}}(A_i) > 0$. For $i \in \Gamma_+$ we equip $\mathbf{A}_i$ with the σ-algebra $\Sigma_{\mathbf{A}_i}$ and the probability measure $\nu_{\mathbf{A}_i}$, where $\Sigma_{\mathbf{A}_i}$ is restriction of $\Sigma_{\mathbf{A}}$ to A_i and, for $X \in \Sigma_{\mathbf{A}_i}$, $\nu_{\mathbf{A}_i}(X) = \nu_{\mathbf{A}}(X)/\nu_{\mathbf{A}}(A_i)$. Then

$$\langle \psi, \mathbf{A}\rangle = \sum_{i\in\Gamma} \nu_{\mathbf{A}}(A_i)^p \, \langle \psi, \mathbf{A}_i\rangle.$$

Proof. First note that each connected component of $\mathbf{A}$ is measurable: let $\mathbf{A}_i$ be a connected component of $\mathbf{A}$ and let $a \in A_i$. Then $A_i = \{x \in A : \mathbf{A} \models \varpi(x,a)\}$ hence A_i is measurable as $\mathbf{A}$ is a relational sample space. Let $Y = \{(v_1,\dots,v_p) \in A^p : \mathbf{A} \models \psi(v_1,\dots,v_p)\}$. Then $\langle \psi, \mathbf{A}\rangle = \nu_{\mathbf{A}}^p(Y)$. As ψ is ϖ-local, it also holds $Y = \bigcup_{i\in\Gamma} Y_i$, where $Y_i = \{(v_1,\dots,v_p) : \mathbf{A}_i \models \psi(v_1,\dots,v_p)\} = Y \cap A_i^p$. As $A_i \in \Sigma_{\mathbf{A}}$ and $Y \in \Sigma_{\mathbf{A}}^p$, it follows that $Y_i \in \Sigma_{\mathbf{A}}^p$ and (by countable additivity) it holds

$$\langle \psi, \mathbf{A}\rangle = \nu_{\mathbf{A}}^p(Y) = \sum_{i\in\Gamma} \nu_{\mathbf{A}}^p(Y_i) = \sum_{i\in\Gamma_+} \nu_{\mathbf{A}}(\mathbf{A}_i)^p \nu_{\mathbf{A}_i}^p(Y_i) = \sum_{i\in\Gamma} \nu_{\mathbf{A}}(\mathbf{A}_i)^p\, \langle \psi, \mathbf{A}_i\rangle.$$

□ □

Corollary 2. *Let $\mathbf{A}$ be a finite λ-structure with component relation ϖ. Let $\psi \in \mathrm{FO}_p(\lambda)$ be a ϖ-local formula of $\mathbf{A}$.*

Let $\mathbf{A}_1,\dots,\mathbf{A}_n$ be the connected components of $\mathbf{A}$. Then

$$\langle \psi, \mathbf{A}\rangle = \sum_{i=1}^n \left(\frac{|A_i|}{|A|}\right)^p \langle \psi, \mathbf{A}_i\rangle.$$

In the aim of extending this reduction result to all local first-order formulas (Theorem 21), we consider, as a first step, the case of finite conjunctions of ϖ-local formulas.

Lemma 26. *Let $\psi_1,\dots,\psi_n$ be ϖ-local formulas.*

Then for modeling $\mathbf{A} \in \mathcal{K}_\varpi$ it holds

$$\langle \bigwedge_{i=1}^n \psi_i, \mathbf{A}\rangle = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{C\in[n]/\sigma} \langle \bigtriangleup_{i\in C}^{\varpi} \psi_i, \mathbf{A}\rangle.$$

Proof. For each $i = 1,\dots,n$ select some x_{a_i} in $\mathrm{Fv}(\psi_i)$. For a partition τ of $[n]$ we denote by ζ_τ the conjunction of $\varpi(x_{a_i}, x_{a_j})$ for every i,j belonging to a same part and of $\neg\varpi(x_{a_i}, x_{a_j})$ for every i,j belonging to different parts. Then it is easily checked that every partition τ of $[n]$ it holds

$$\prod_{C\in\tau} \langle \bigtriangleup_{i\in C}^{\varpi} \psi_i, \mathbf{A}\rangle = \sum_{\tau'\geq\tau} \langle \zeta_\tau \wedge \bigwedge_{i=1}^n \psi_i, \mathbf{A}\rangle.$$

Recall that the Möbius function of the lattice of the partitions of $[n]$ is

$$\mu(\tau,\tau') = (-1)^{|\tau|-|\tau'|} \prod_{i=3}^n ((i-1)!)^{r_i},$$

where $|\tau|$ is the number of parts of τ and r_i is the number of parts of τ' that are unions of i parts of τ.

Moreover, to every permutation $\sigma \in \mathfrak{S}_n$ we can associate the partition $[n]/\sigma$. Note that a partition τ of $[n]$ is obtained from exactly $\prod_{i=3}^n ((i-1)!)^{n_i}$ permutations where n_i is the number of parts of τ with exactly n_i elements (see for instance [7, 63]).

By Möbius inversion, denoting by τ_0 the trivial partition with all parts of size 1, it holds

$$\begin{aligned}\langle \bigwedge_{i=1}^n \psi_i, \mathbf{A}\rangle &= \sum_{\tau} (-1)^{|\tau_0|-|\tau|} \mu(\tau_0,\tau) \prod_{C\in\tau} \langle \bigtriangleup_{i\in C}^{\varpi} \psi_i, \mathbf{A}\rangle \\ &= \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{C\in[n]/\sigma} \langle \bigtriangleup_{i\in C}^{\varpi} \psi_i, \mathbf{A}\rangle.\end{aligned}$$

□ □

We are now ready to reduce Stone pairing of local formulas to Stone pairings with ϖ-local formulas.

Lemma 27. *Let $\phi \in \mathrm{FO}_p^{\mathrm{local}}(\lambda)$. Then there exist, for every partition τ of $[p]$ with parts $I_1, \dots, I_{|\tau|}$, ϖ-local formulas $\phi_{\tau,i}$ (with $1 \leq i \leq |\tau|$) with free variables x_j (for $j \in I_i$) and constants c_i such that for modeling $\mathbf{A} \in \mathcal{K}_\varpi$ it holds*

$$\langle \phi, \mathbf{A}\rangle = \sum_\tau \sum_{c \in C_\tau} \sum_{\sigma \in \mathfrak{S}_{|\tau|}} (-1)^{\epsilon(\sigma)} \prod_{P \in [|\tau|]/\sigma} \langle \bigtriangleup^{\varpi}_{i \in P} (\phi_{\tau,i} \oplus c_i), \mathbf{A}\rangle.$$

Proof. For a partition τ of $[p]$ we denote by ζ_τ the conjunction of $\varpi(x_i, x_j)$ for every i, j belonging to a same part and of $\neg\varpi(x_i, x_j)$ for every i, j belonging to different parts. Then, for any two distinct partitions τ and τ', the formula $\zeta_\tau \wedge \zeta'_\tau$ is never satisfied; moreover $\bigvee_\tau \zeta_\tau$ is always satisfied. Thus

$$\phi = \bigvee_\tau (\zeta_\tau \wedge \phi)$$

and, as the terms of the disjunction are mutually exclusive, it holds

$$\langle \phi, \mathbf{A}\rangle = \sum_\tau \langle \zeta_\tau \wedge \phi, \mathbf{A}\rangle.$$

For every partition τ with parts $I_1, \dots, I_{|\tau|}$ there exist ϖ-local formulas $\phi_{\tau,i}$ (with $1 \leq i \leq |\tau|$) with free variables x_j (for $j \in I_i$) and a Boolean function g_τ, such that it holds

$$\mathbf{A} \models \zeta_\tau \wedge \phi \quad\iff\quad \mathbf{A} \models g_\tau(\phi_{\tau,1}, \dots, \phi_{\tau,|\tau|}).$$

Denote by $\oplus$ the exclusive disjunction, so that for every ψ it holds $\psi \oplus 0 = \psi$ and $\psi \oplus 1 = \neg\psi$. By considering the truth-table of g_τ we can expand $g_\tau(\phi_{\tau,1}, \dots, \phi_{\tau,|\tau|})$ as

$$\bigvee_{c \in C_\tau} \bigwedge_{i=1}^{|\tau|} (\phi_{\tau,i} \oplus c_i),$$

where C_τ is a subset of $\{0,1\}^{|\tau|}$. Note that the terms of the disjunction are mutually exclusive and that $\neg\phi_{\tau,i}$ is ϖ-local, as $\phi_{\tau,i}$ is ϖ-local. Hence

$$\langle \phi, \mathbf{A}\rangle = \sum_\tau \sum_{c \in C_\tau} \langle \bigwedge_{i=1}^{|\tau|} (\phi_{\tau,i} \oplus c_i), \mathbf{A}\rangle.$$

According to Lemma 26, it holds

$$\langle \bigwedge_{i=1}^{|\tau|} (\phi_{\tau,i} \oplus c_i), \mathbf{A}\rangle = \sum_{\sigma \in \mathfrak{S}_{|\tau|}} (-1)^{\epsilon(\sigma)} \prod_{P \in [|\tau|]/\sigma} \langle \bigtriangleup^{\varpi}_{i \in P} (\phi_{\tau,i} \oplus c_i), \mathbf{A}\rangle.$$

Hence

$$\langle \phi, \mathbf{A}\rangle = \sum_\tau \sum_{c \in C_\tau} \sum_{\sigma \in \mathfrak{S}_{|\tau|}} (-1)^{\epsilon(\sigma)} \prod_{P \in [|\tau|]/\sigma} \langle \bigtriangleup^{\varpi}_{i \in P} (\phi_{\tau,i} \oplus c_i), \mathbf{A}\rangle.$$

□ □

We can deduce the generalization of Lemma 25 to local formulas.

Theorem 21. *Let $p \in \mathbb{N}$ and $\phi \in \mathrm{FO}_p^{\mathrm{local}}(\lambda)$. There is an integer s, finite sets I_i (for $1 \leq i \leq s$), values $\epsilon_i \in \{-1, 1\}$ (for $1 \leq i \leq s$), and formulas $\varphi_{i,j} \in \mathrm{FO}_p^{\mathrm{local}}$ (for $1 \leq i \leq s$ and $j \in I_i$) such that for every modeling $\mathbf{A}$ with component relation ϖ, and countable set of connected components $\{\mathbf{A}_k\}_{k \in \Gamma}$, it holds*

$$\langle \phi, \mathbf{A} \rangle = \sum_{i=1}^{s} \epsilon_i \prod_{j \in I_i} \sum_{k \in \Gamma} \nu_{\mathbf{A}}(A_k)^p \langle \varphi_{i,j}, \mathbf{A}_k \rangle.$$

Proof. This is a direct consequence of Lemma 27 and Lemma 25. □ □

The case of sentences can be handled easily. For a set X and an integer m, define

$$\mathrm{Big}_m(X) = \begin{cases} 1 & \text{if } |X| \geq m \\ 0 & \text{otherwise} \end{cases}$$

Lemma 28. *Let $\theta \in \mathrm{FO}_0(\lambda)$.*

Then there exist formulas $\psi_1, \dots, \psi_s \in \mathrm{FO}_1^{\mathrm{local}}$ with quantifier rank at most $q(\mathrm{qrank}(\theta))$, integers $m_1, \dots, m_s \leq \mathrm{qrank}(\theta)$, and a Boolean function F such that for every λ-structure $\mathbf{A}$ with component relation ϖ and connected components $\mathbf{B}_i$ $(i \in I)$, the property $\mathbf{A} \models \theta$ is equivalent to

$$F(\mathrm{Big}_{m_1}(\{i, \mathbf{B}_i \models (\exists x)\psi_1(x)\}), \dots, \mathrm{Big}_{m_s}(\{i, \mathbf{B}_i \models (\exists x)\psi_s(x)\})) = 1.$$

Proof. Indeed, it follows from Gaifman locality theorem 10 that — in presence of a component relation ϖ — every sentence θ with quantifier rank r can be written as a Boolean combination of sentences θ_k of the form

$$\exists y_1 \dots \exists y_{m_k} \left(\bigwedge_{1 \leq i < j \leq m_k} \neg\varpi(y_i, y_j) \wedge \bigwedge_{1 \leq i \leq m_k} \psi_k(y_i) \right)$$

where ψ_k is ϖ-local, $m_k \leq \mathrm{qrank}(\theta)$, and $\mathrm{qrank}(\psi_k) \leq q(\mathrm{qrank}(\theta))$, for some fixed function q. As $\mathbf{A} \models \theta_k$ if and only if $\mathrm{Big}_{m_k}(\{i, \mathbf{B}_i \models (\exists x)\psi_k(x)\}) = 1$, the lemma follows. □ □

8.3. **Convex Combinations of Modelings.** In several contexts, it is clear when disjoint union of converging sequences form a converging sequence. If two graph sequences $(G_n)_{n \in \mathbb{N}}$ and $(H_n)_{n \in \mathbb{N}}$ are L-convergent or BS-convergent, it is clear that the sequence $(G_n \cup H_n)_{n \in \mathbb{N}}$ is also convergent, provided that the limit

$$\lim_{n \to \infty} |G_n|/(|G_n| + |H_n|)$$

exists. The same applies if we merge a countable set of L-convergent (resp. BS-convergent) sequences $(H_{n,i})_{n \in \mathbb{N}}$ (where $i \in \mathbb{N}$), with the obvious restriction that for each $i \in \mathbb{N}$ all but finitely many $H_{n,i}$ are empty graphs.

We shall see that the possibility to merge up to a countable set of converging sequences to $\mathrm{FO}^{\mathrm{local}}$-convergence will need a further assumption, namely the following equality:

$$\sum_i \lim_{n \to \infty} \frac{|G_{n,i}|}{|\bigcup_j G_{n,j}|} = 1.$$

The importance of this assumption is illustrated by the next example.

Example 8. Let $N_n = 2^{2^n}$ (so that $N(n)$ is divisible by 2^i for every $1 \leq i \leq 2^n$). Consider sequences $(H_{n,i})_{n\in\mathbb{N}}$ of edgeless black and white colored graphs where $H_{n,i}$ is

- empty if $i > 2^n$,
- the edgeless graph with $(2^{-i} + 2^{-n})N_n$ white vertices and $2^{-i}N_n$ black vertices if n is odd,
- the edgeless graph with $(2^{-i} + 2^{-n})N_n$ black vertices and $2^{-i}N_n$ white vertices if n is even.

For each $i \in \mathbb{N}$, the sequence $(H_{n,i})_{n\in\mathbb{N}}$ is obviously L-convergent (and even FO-convergent) as the proportion of white vertices in $H_{n,i}$ tends to $1/2$ as $n \to \infty$. The order of $G_n = \bigcup_{i\in\mathbb{N}} H_{n,i}$ is $3N_n$ and $|H_{n,i}|/|G_n|$ tends to $\frac{2}{3}.2^{-i}$ as n goes to infinity. However the sequence $(G_n)_{n\in\mathbb{N}}$ is not L-convergent (hence not $\mathrm{FO}^{\mathrm{local}}$-convergent). Indeed, the proportion of white vertices in G_n is $2/3$ if n is odd and $1/3$ is n is even.

Definition 14 (Convex combination of Modelings)**.** Let $\mathbf{H}_i$ be λ-modelings for $i \in I \subseteq \mathbb{N}$ and let $(\alpha_i)_{i\in I}$ be positive real numbers such that $\sum_{i\in I} \alpha_i = 1$.

Let $\mathbf{H}$ be the disjoint union of the $\mathbf{H}_i$, let $\Sigma_{\mathbf{H}} = \{\bigcup_i X_i : \ X_i \in \Sigma_{\mathbf{H}_i}\}$ and, for $X \in \Sigma_{\mathbf{H}}$, let $\nu_{\mathbf{H}}(X) = \sum_i \alpha_i \nu_{\mathbf{H}_i}(X \cap H_i)$.

Then $\mathbf{H}$ is the *convex combination* of modelings $\mathbf{H}_i$ with *weights* α_i and we denote it by $\coprod_{i\in I}(\mathbf{H}_i, \alpha_i)$.

Lemma 29. *Let $\mathbf{H}_i$ be λ-modelings for $i \in I \subseteq \mathbb{N}$ and let $(\alpha_i)_{i\in I}$ be positive real numbers such that $\sum_{i\in I} \alpha_i = 1$. Let $\mathbf{H} = \coprod_{i\in I}(\mathbf{H}_i, \alpha_i)$ Then*

(1) $\mathbf{H}$ is a modeling, each H_i is measurable and $\nu_{\mathbf{H}}(H_i) = \alpha_i$ holds for every $i \in I$;
(2) if all the $\mathbf{H}_i$ are weakly uniform and either all the H_i are infinite or all the H_i are finite, I is finite, and $\alpha_i = |H_i|/\sum_{i\in I}|H_i|$, then $\mathbf{H}$ is weakly uniform.

Proof. We consider the signature λ^+ obtained from λ by adding a new binary relation ϖ, and the basic interpretation scheme I_1 of λ^+-structures in λ-structures corresponding to the addition of the new relation ϖ by the formula $\theta_\varpi = 1$. This means that for every λ-structure $\mathbf{A}$ it holds $\mathsf{I}_1(\mathbf{A}) \models \forall x, y\ \varpi(x, y)$. Let $\mathbf{H}_i^+ = \mathsf{I}_1(\mathbf{H}_i)$. This is a weakly uniform modeling.

Let $\mathbf{H}^+ = \coprod_{i\in I}(\mathbf{H}_i^+, \alpha_i)$. Clearly, the family $\Sigma_{\mathbf{H}^+}$ is a σ-algebra, $(H, \Sigma_{\mathbf{H}^+})$ is a standard Borel space, and $\nu_{\mathbf{H}^+}$ is a probability measure. Moreover, by construction, H_i^+ is measurable and $\nu_{\mathbf{H}^+}(H_i^+) = \alpha_i$.

Let $\phi \in \mathrm{FO}_p(\lambda)$. First notice that for every $(v_1, \dots, v_p) \in H^{p+q}$ (which is also $(H^+)^{p+q}$) it holds $\Omega_\phi(\mathbf{H}) = \Omega_\phi(\mathbf{H}^+)$, that is:

$$\mathbf{H} \models \phi(v_1, \dots, v_p) \quad \iff \quad \mathbf{H}^+ \models \phi(v_1, \dots, v_p).$$

It follows from Lemma 27 that the set $\Omega_\phi(\mathbf{H}^+)$ may be obtained as a Boolean combination of products of sets defined by ϖ-local formulas. So, we can assume that ϕ is ϖ-local. Then $\Omega_\phi(\mathbf{H}^+)$ is the union of the sets $\Omega_\phi(\mathbf{H}_i)$. All these sets are measurable (as $\mathbf{H}_i$ is a modeling) thus their union is measurable (by construction of $\Sigma_{\mathbf{H}}$). It follows that $\mathbf{H}^+$ is a modeling, and so is $\mathbf{H} = \mathbf{H}^+ - \varpi$.

Assume that all the $\mathbf{H}_i$ are weakly uniform. If all the $\mathbf{H}_i$ are finite, I is finite, and $\alpha_i = |H_i|/\sum_{i\in I}|H_i|$, then $\mathbf{H}$ is the modeling associated to the union of the $\mathbf{H}_i$ hence it is weakly uniform. Otherwise all the H_i are infinite, hence all the $\nu_{\mathbf{H}_i}$ are atomless, $\nu_{\mathbf{H}}$ is atomless, and $\mathbf{H}$ is weakly uniform. □ □

Lemma 30. *Let $p \in \mathbb{N}$ and $\phi \in \mathrm{FO}_p^{\mathrm{local}}(\lambda)$. There is an integer s, finite sets I_i (for $1 \leq i \leq s$), values $\epsilon_i \in \{-1, 1\}$ (for $1 \leq i \leq s$), and formulas $\varphi_{i,j} \in \mathrm{FO}_p^{\mathrm{local}}$*

(for $1 \le i \le s$ and $j \in I_i$) such that for every countable set of modelings $\mathbf{A}_j$ and weights α_j ($j \in J \subseteq \mathbb{N}$ and $\sum_j \alpha_j = 1$) it holds

$$\langle \phi, \mathbf{A} \rangle = \sum_{i=1}^{s} \epsilon_i \prod_{j \in I_i} \sum_{k \in \Gamma} \alpha_k^p \langle \varphi_{i,j}, \mathbf{A}_k \rangle.$$

Proof. Considering, as above, the combination $\mathbf{H}^+ = \coprod_{i\in I}(\mathbf{H}_i^+, \alpha_i)$, where $\mathbf{H}_i^+$ is obtained by the basic interpretation scheme adding a full binary relation ϖ, the result is an immediate consequence of Theorem, 21. □ □

Theorem 22. *Let $p \in \mathbb{N}$, let $I \subseteq \mathbb{N}$ and, for each $i \in I$ let $(\mathbf{A}_{i,n})_{n\in\mathbb{N}}$ be an $\mathrm{FO}_p^{\mathrm{local}}(\lambda)$-convergent sequence of λ-modelings and let $(a_{i,n})_{n\in\mathbb{N}}$ be a convergent sequence of non-negative real numbers, such that $\sum_{i\in I} a_{i,n} = 1$ holds for every $n \in \mathbb{N}$, and such that $\sum_{i\in I} \lim_{n\to\infty} a_{i,n} = 1$.*

Then the sequence of convex combinations $\coprod_{i\in I}(\mathbf{A}_{i,n}, a_{i,n})$ is $\mathrm{FO}_p^{\mathrm{local}}(\lambda)$-convergent.

Proof. If I is finite, then the result follows from Lemma 30. Hence we can assume $I = \mathbb{N}$.

Let $\phi \in \mathrm{FO}_p^{\mathrm{local}}$, let $q \in \mathbb{N}$, and let $\epsilon > 0$ be a positive real. Assume that for each $i \in \mathbb{N}$ the sequence $(\mathbf{A}_{i,n})_{n\in\mathbb{N}}$ is $\mathrm{FO}_p^{\mathrm{local}}$-convergent and that $(a_{i,n})_{n\in\mathbb{N}}$ is a convergent sequence of non-negative real numbers, such that $\sum_i a_{i,n} = 1$ holds for every $n \in \mathbb{N}$. Let $\alpha_i = \lim_{n\to\infty} a_{i,n}$, let $d_i = \lim_{n\to\infty}\langle \phi, \mathbf{A}_{i,n}\rangle$, and let C be such that $\sum_{i=1}^{C} \alpha_i > 1 - \epsilon/4$. There exists N such that for every $n \ge N$ and every $i \le C$ it holds $|a_{n,i} - \alpha_i| < \epsilon/4C$ and $|a_{i,n}^q \langle \phi, \mathbf{A}_{i,n}\rangle - \alpha_i^q d_i| < \epsilon/2C$. Thus $\left|\sum_{i=1}^{C} a_{i,n}^q \langle \phi, \mathbf{A}_{i,n}\rangle - \sum_{i=1}^{C} \alpha_i^q d_i\right| < \epsilon/2$ and $\sum_{i>C+1} a_{i,n} < \epsilon/2$. It follows that for any $n \ge N$ it holds

$$\left|\sum_{i>C+1} a_{i,n}^q \langle \phi, \mathbf{A}_{i,n}\rangle - \sum_{i>C+1} \alpha_i^q d_i\right| \le \max\left(\sum_{i>C+1} a_{i,n}^q, \sum_{i>C+1} \alpha_i^q d_i\right) < \epsilon/2$$

hence $|\sum_i a_{i,n}^q \langle \phi, \mathbf{A}_{i,n}\rangle - \sum_i \alpha_i^q d_i| < \epsilon$.

For every $\psi \in \mathrm{FO}_p^{\mathrm{local}}$, the expression appearing in Lemma 30 for the expansion of $\langle \phi, \coprod_i(\mathbf{A}_{i,n}, a_{i,n})\rangle$ is a finite combination of terms of the form $\sum_i a_{i,n}^q \langle \phi, \mathbf{A}_{i,n}\rangle$, where $q \in \mathbb{N}$ and $\phi \in \mathrm{FO}_p^{\mathrm{local}}$. It follows that the value $\langle \phi, \coprod_i(\mathbf{A}_{i,n}, a_{i,n})\rangle$ converges as n grows to infinity. Hence $(\coprod_i(\mathbf{A}_{i,n}, a_{i,n}))_{n\in\mathbb{N}}$ is $\mathrm{FO}_p^{\mathrm{local}}$-convergent. □ □

Corollary 3. *Let $p \ge 1$ and let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a sequence of finite λ-structures.*

Assume $\mathbf{A}_n$ be the disjoint union of $\mathbf{B}_{n,i}$ ($i \in \mathbb{N}$) where all but a finite number of $\mathbf{B}_{n,i}$ are empty. Let $a_{n,i} = |B_{n,i}|/|A_n|$. Assume further that:

- *for each $i \in \mathbb{N}$, the limit $\alpha_i = \lim_{n\to\infty} a_{n,i}$ exists,*
- *for each $i \in \mathbb{N}$ such that $\alpha_i \neq 0$, the sequence $(\mathbf{B}_{n,i})_{n\in\mathbb{N}}$ is $\mathrm{FO}_p^{\mathrm{local}}$-convergent,*
- *it holds*
$$\sum_{i\ge 1} \alpha_i = 1.$$

Then, the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_p^{\mathrm{local}}$-convergent.

Moreover, if $\mathbf{L}_i$ is a modeling $\mathrm{FO}_p^{\mathrm{local}}$-limit of $(\mathbf{B}_{n,i})_{n\in\mathbb{N}}$ when $\alpha_i \neq 0$ then $\coprod_i(\mathbf{L}_i, \alpha_i)$ is a modeling $\mathrm{FO}_p^{\mathrm{local}}$-limit of $(\mathbf{A}_n)_{n\in\mathbb{N}}$.

Proof. This follows from Theorem 22, as $\mathbf{A}_n = \coprod_i (\mathbf{B}_{n,i}, a_{n,i})$. □ □

Definition 15. A family of sequence $(\mathbf{A}_{i,n})_{n\in\mathbb{N}}$ $(i \in I)$ of λ-structures is *uniformly elementarily convergent* if, for every formula $\phi \in \mathrm{FO}_1(\lambda)$ there is an integer N such that it holds

$$\forall i \in I,\ \forall n' \geq n \geq N, \quad (\mathbf{A}_{i,n} \models (\exists x)\phi(x)) \Longrightarrow (\mathbf{A}_{i,n'} \models (\exists x)\phi(x)).$$

First notice that if a family $(\mathbf{A}_{i,n})_{n\in\mathbb{N}}$ $(i \in I)$ of sequences is uniformly elementarily convergent, then each sequence $(\mathbf{A}_{i,n})_{n\in\mathbb{N}}$ is elementarily convergent

Lemma 31. *Let $I \subseteq N$, and let $(\mathbf{A}_{i,n})_{n\in\mathbb{N}}$ $(i \in I)$ be sequences forming a uniformly elementarily convergent family.*

Then $(\bigcup_{i\in I} \mathbf{A}_{i,n})_{n\in\mathbb{N}}$ is elementarily convergent.

Moreover, if $(\mathbf{A}_{i,n})_{n\in\mathbb{N}}$ is elementarily convergent to $\widehat{\mathbf{A}}_i$ then $(\bigcup_{i\in I} \mathbf{A}_{i,n})_{n\in\mathbb{N}}$ is elementarily convergent to $\bigcup_{i\in I} \widehat{\mathbf{A}}_i$.

Proof. Let λ^+ be the signature λ augmented by a binary relational symbol ϖ. Let I_1 be the basic interpretation scheme of λ^+-structures in λ-structures defining $\varpi(x, y)$ for every x, y. Let $\mathbf{A}^+_{i,n} = I_1(\mathbf{A}_{i,n})$. According to Lemma 28, for every sentence $\theta \in \mathrm{FO}_0(\lambda)$ there exist formulas $\psi_1, \dots, \psi_s \in \mathrm{FO}_1^{\mathrm{local}}$, an integer m, and a Boolean function F such that the property $\bigcup_{i\in I} \mathbf{A}^+_{i,n} \models \theta$ is equivalent to

$$F(\mathrm{Big}_{m_1}(\{i, \mathbf{A}_{i,n} \models (\exists x)\psi_1(x)\}), \dots, \mathrm{Big}_{m_s}(\{i, \mathbf{A}_{i,n} \models (\exists x)\psi_s(x)\})) = 1.$$

According to the definition of a uniformly elementarily convergent family there is an integer N such that, for every $1 \leq j \leq s$, the value $\mathrm{Big}_{m_j}(\{i, \mathbf{A}_{i,n} \models (\exists x)\psi_j(x)\})$ is a function of n, which is non-decreasing for $n \geq N$. It follows that this function admits a limit for every $1 \leq j \leq s$ hence the exists an integer N' such that either $\bigcup_{i\in I} \mathbf{A}^+_{i,n} \models \theta$ holds for every $n \geq N'$ or it holds for no $n \geq N'$. It follows that $(\bigcup_{i\in I} \mathbf{A}^+_{i,n})_{n\in\mathbb{N}}$ is elementarily convergent. Thus (by means of the basic interpretation scheme deleting ϖ) $(\bigcup_{i\in I} \mathbf{A}_{i,n})_{n\in\mathbb{N}}$ is elementarily convergent

If I is finite, it is easily checked that if $(\mathbf{A}_{i,n})_{n\in\mathbb{N}}$ is elementarily convergent to $\widehat{\mathbf{A}}_i$ then $(\bigcup_{i\in I} \mathbf{A}_{i,n})_{n\in\mathbb{N}}$ is elementarily convergent to $\bigcup_{i\in I} \widehat{\mathbf{A}}_i$.

Otherwise, we can assume $I = \mathbb{N}$. Following the same lines, it is easily checked that $(\bigcup_{i=1}^n \widetilde{\mathbf{A}}_i)_{n\in\mathbb{N}}$ converges elementarily to $(\bigcup_{i\in\mathbb{N}} \widetilde{\mathbf{A}}_i)_{n\in\mathbb{N}}$. For $i, n \in \mathbb{N}$, let $\mathbf{B}_{i,2n} = \mathbf{A}_{i,n}$ and $\mathbf{B}_{i,2n+1} = \widetilde{\mathbf{A}_i}$. As, for each $i \in \mathbb{N}$, $\widetilde{\mathbf{A}}_i$ is an elementary limit of $(\mathbf{A}_{i,n})_{n\in\mathbb{N}}$ it is easily checked that the family of the sequences $(\mathbf{B}_{i,n})_{n\in\mathbb{N}}$ is uniformly elementarily convergent. It follows that $(\bigcup_{i\in\mathbb{N}} \mathbf{B}_{i,n})_{n\in\mathbb{N}}$ is elementarily convergent thus the elementary limit of $(\bigcup_{i\in I} \mathbf{A}_{i,n})_{n\in\mathbb{N}}$ and $(\bigcup_{i=1}^n \widetilde{\mathbf{A}}_i)_{n\in\mathbb{N}}$ are the same, that is $\bigcup_{i\in I} \widetilde{\mathbf{A}}_i$. □ □

From Corollary 3 and Lemma 31 then follows the next general result.

Corollary 4. *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a sequence of finite λ-structures.*

Assume $\mathbf{A}_n$ be the disjoint union of $\mathbf{B}_{n,i}$ $(i \in \mathbb{N})$ where all but a finite number of $\mathbf{B}_{n,i}$ are empty. Let $a_{n,i} = |B_{n,i}|/|A_n|$. Assume that:

- *for each $i \in \mathbb{N}$, the limit $\alpha_i = \lim_{n\to\infty} a_{n,i}$ exists and it holds*
$$\sum_{i\geq 1} \alpha_i = 1,$$
- *for each $i \in \mathbb{N}$ such that $\alpha_i \neq 0$, the sequence $(\mathbf{B}_{n,i})_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent,*
- *the family $\{(\mathbf{B}_{n,i})_{n\in\mathbb{N}}\ (i \in \mathbb{N})\}$ is uniformly elementarily convergent.*

Then, the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *is* FO*-convergent.*

Moreover, if $\mathbf{L}_i$ *is a modeling* FO*-limit of* $(\mathbf{B}_{n,i})_{n\in\mathbb{N}}$ *when* $\alpha_i \neq 0$ *and an elementary limit of* $(\mathbf{B}_{n,i})_{n\in\mathbb{N}}$ *when* $\alpha_i = 0$ *then* $\coprod_i(\mathbf{L}_i, \alpha_i)$ *is a modeling* FO*-limit of* $(\mathbf{A}_n)_{n\in\mathbb{N}}$.

9. Decomposing Sequences: the Comb Structure

Definition 16. Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a sequence of finite λ-structures, having $\varpi \in \lambda$ as a component relation. In the following, we assume that ϖ-spectra are extended to infinite sequences by adding zeros if necessary.

- The sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is ϖ*-nice* if $\mathrm{Sp}_\varpi(\mathbf{A}_n)$ converges pointwise;
- The *limit* ϖ*-spectrum* of a ϖ-nice sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is the pointwise limit of $\mathrm{Sp}_\varpi(\mathbf{A}_n)$;
- the ϖ*-support* is the set I of the indexes i for which the limit ϖ-spectrum is non-zero;
- the sequence has *full* ϖ-spectrum if, for every index i not in the ϖ-support, there is some N such that the ith value of $\mathrm{Sp}_\varpi(\mathbf{A}_n)$ is zero for every $n > N$.

As proved in Lemma 24, every $\mathrm{FO}^{\mathrm{local}}$-convergent sequence is ϖ-nice.

Lemma 32. *Let* $(\mathbf{A}_n)$ *be a* ϖ*-nice sequence of* λ*-structures with empty* ϖ*-support. Then the following conditions are equivalent:*

(1) the sequence $(\mathbf{A}_n)$ *is* $\mathrm{FO}^{\mathrm{local}}$*-convergent;*

(2) the sequence $(\mathbf{A}_n)$ *is* $\mathrm{FO}_1^{\mathrm{local}}$*-convergent.*

Moreover, for every ϖ*-local formula* ϕ *with* $p > 1$ *free variables it holds*

$$\lim_{n\to\infty} \langle \phi, \mathbf{A}_n \rangle = 0.$$

Proof. $\mathrm{FO}^{\mathrm{local}}$-convergence obviously implies $\mathrm{FO}_1^{\mathrm{local}}$-convergence. So, assume that $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent, and let ϕ be a ϖ-local first-order formula with $p > 1$ free variables. For $n \in \mathbb{N}$, let $\mathbf{B}_{n,i}$ $(i \in \Gamma_n)$ denote the connected components of $\mathbf{A}_n$. As $(\mathbf{A}_n)$ is ϖ-nice and has empty ϖ-support, there exists for every $\epsilon > 0$ an integer N such that for $n > N$ and every $i \in \Gamma_n$ it holds $|B_{n,i}| < \epsilon |A_n|$. Then, according to Corollary 2, for $n > N$

$$\begin{aligned}
\langle \phi, \mathbf{A}_n \rangle &= \sum_{i\in\Gamma_n} \left(\frac{|B_{n,i}|}{|A_n|}\right)^p \langle \phi, B_{n,i} \rangle \\
&\leq \sum_{i\in\Gamma_n} \left(\frac{|B_{n,i}|}{|A_n|}\right)^p \\
&< \sum_{i\in\Gamma_n} \frac{|B_{n,i}|}{|A_n|} \epsilon^{p-1} = \epsilon^{p-1}
\end{aligned}$$

Hence $\langle \phi, \mathbf{A}_n \rangle$ converges (to 0) as n grows to infinity. It follows that $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent, according to Lemma 27. □ □

Lemma 33. *Let* $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *be an* $\mathrm{FO}^{\mathrm{local}}$*-convergent sequence of finite* λ*-structures, with component relation* ϖ *and limit* ϖ*-spectrum* $(\lambda_i)_{i\in I}$*. For* $n \in \mathbb{N}$*, let* $\mathbf{B}_{n,i}$ *be the connected components of* $\mathbf{A}_n$ *order in non-decreasing order (with* $\mathbf{B}_{n,i}$ *empty if* i *is greater than the number of connected components of* $\mathbf{A}_n$*). Let* $a \leq b$ *be the first and last occurrence of* $\lambda_a = \lambda_b$ *in the* ϖ*-spectrum and let* $\mathbf{A}'_n$ *be the union of all the* $\mathbf{B}_{n,i}$ *for* $a \leq i \leq b$*.*

Then $(\mathbf{A}'_n)_{n\in\mathbb{N}}$ *is* FO*-convergent if* $\lambda_a > 0$ *and* $\mathrm{FO}^{\mathrm{local}}$*-convergent if* $\lambda_a = 0$*.*

Assume moreover that $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *has a modeling* $\mathrm{FO}^{\mathrm{local}}$*-limit* $\mathbf{L}$*. Let* $\mathbf{L}'$ *be the union of the connected components* $\mathbf{L}_i$ *of* $\mathbf{L}$ *with* $\nu_{\mathbf{L}}(L_i)=\lambda_a$*. Equip* $\mathbf{L}'$ *with the* σ*-algebra* $\Sigma_{\mathbf{L}'}$ *which is the restriction of* $\Sigma_{\mathbf{L}}$ *to* L' *and the probability measure* $\nu_{\mathbf{L}'}$ *defined by* $\nu_{\mathbf{L}'}(X)=\nu_{\mathbf{L}}(X)/\nu_{\mathbf{L}}(L')$ *(for* $X\in\Sigma_{\mathbf{L}'}$*).*

Then $\mathbf{L}'$ *is a modeling* FO*-limit of* $(\mathbf{A}'_n)_{n\in\mathbb{N}}$ *if* $\lambda_a>0$ *and a modeling* $\mathrm{FO}^{\mathrm{local}}$*-limit of* $(\mathbf{A}'_n)_{n\in\mathbb{N}}$ *if* $\lambda_a=0$*.*

Proof. Extend the sequence λ to the null index by defining $\lambda_0=2$. Let $r=\min(\lambda_{a-1}/\lambda_a,\lambda_b/\lambda_{b+1})$ (if $\lambda_{b+1}=0$ simply define $r=\lambda_{a-1}/\lambda_a$). Notice that $r>1$. Let ϕ be a ϖ-local formula with p free variables. According to Corollary 2 it holds

$$\langle\phi,\mathbf{A}_n\rangle=\sum_i\left(\frac{|B_{n,i}|}{|A_n|}\right)^p\langle\phi,\mathbf{B}_{n,i}\rangle.$$

In particular, it holds

$$\langle\varpi^{(p)},\mathbf{A}_n\rangle=\sum_i\left(\frac{|B_{n,i}|}{|A_n|}\right)^p.$$

Let $\alpha>1/(1-r^p)$. Define

$$w_{n,i}=\left(\frac{|B_{n,i}|}{|A_n|}\right)^p(\alpha+\langle\phi,\mathbf{B}_{n,i}\rangle).$$

From the definition of r it follows that for each $n\in\mathbb{N}$, $w_{n,i}>w_{n,j}$ if $i<a$ and $j\geq a$ or $i\leq b$ and $j>b$. Let $\sigma\in\mathfrak{S}_\infty$ be such that $a_{n,i}=w_{n,\sigma(i)}$ is non-increasing. It holds

$$\sum_i a_{n,i}=\sum_i w_{n,i}=\alpha\langle\varpi^{(p)},\mathbf{A}_n\rangle+\langle\phi,\mathbf{A}_n\rangle.$$

Hence

$$s_p=\lim_{n\to\infty}\sum_i a_{n,i}^p$$

exists. According to Lemma 23 it follows that for every $i\in\mathbb{N}$ the limit $\lim_{n\to\infty}a_{n,i}$ exists. Moreover, as σ globally preserves the set $\{a,\dots,b\}$ it follows that the limit

$$d=\lim_{n\to\infty}\sum_{i=a}^b\left(\frac{|B_{n,i}|}{|A_n|}\right)^p(\alpha+\langle\phi,\mathbf{B}_{n,i}\rangle)$$

exists. As for every $i\in\{a,\dots,b\}$ it holds $\lim_{n\to\infty}|B_{n,i}|/|A_n|=\lambda_a$ and as $\langle\phi,\mathbf{A}'_n\rangle=\sum_{i=a}^b(|B_{n,i}|/|A_n|)^p\langle\phi,\mathbf{B}_{n,i}\rangle$ we deduce

$$\lim_{n\to\infty}\langle\phi,\mathbf{A}'_n\rangle=d-(b-a+1)\alpha.$$

Hence $\lim_{n\to\infty}\langle\phi,\mathbf{A}'_n\rangle$ exists for every ϖ-local formula and, according to Lemma 27, the sequence $(\mathbf{A}'_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent.

Assume $\lambda>0$. Let $N=b-a+1$. To each sentence θ we associate the formula $\widetilde{\theta}\in\mathrm{FO}_N^{\mathrm{local}}$ that asserts that the substructure induced by the closed neighborhood of $x_1,\dots,x_N$ satisfies θ and that $x_1,\dots,x_N$ are pairwise distinct and non-adjacent. For sufficiently large n, the structure $\mathbf{A}'_n$ has exactly N connected components. It is easily checked that if $\mathbf{A}'_n$ does not satisfy θ then $\langle\widetilde{\theta},\mathbf{A}'_n\rangle=0$, although if $\mathbf{A}'_n$ does not satisfy θ then

$$\langle\widetilde{\theta},\mathbf{A}'_n\rangle\geq\left(\frac{\min_{a\leq i\leq b}|B_{n,i}|}{\sum_{i=a}^b|B_{n,i}|}\right)^N,$$

hence $\langle\widetilde{\theta},\mathbf{A}'_n\rangle>(2N)^{-N}$ for all sufficiently large n. As $\langle\widetilde{\theta},\mathbf{A}'_n\rangle$ converges for every sentence θ, we deduce that the sequence $(\mathbf{A}'_n)_{n\in\mathbb{N}}$ is elementarily convergent. According to Theorem 11, the sequence $(\mathbf{A}'_n)_{n\in\mathbb{N}}$ is thus FO-convergent.

Now assume that $(\mathbf{A}_n)_{n\in\mathbb{N}}$ has a modeling $\mathrm{FO}^{\mathrm{local}}$-limit $\mathbf{L}$. First note that L_i being an equivalence class of ϖ it holds $L_i \in \Sigma_{\mathbf{L}}$, hence $L' \in \Sigma_{\mathbf{L}}$ and $\nu_{\mathbf{L}}(L')$ is well defined. For every ϖ-local formula $\phi \in \mathrm{FO}_p(\lambda)$ it holds, according to Lemma 25:

$$\begin{aligned}\langle \phi, \mathbf{L}'\rangle &= \sum_{i=a}^{b} \nu_{\mathbf{L}'}(L_i)^p \langle \phi, \mathbf{L}_i\rangle \\ &= \frac{1}{\nu_{\mathbf{L}}(L')^p} \sum_{i=a}^{b} \nu_{\mathbf{L}}(L_i)^p \langle \phi, \mathbf{L}_i\rangle\end{aligned}$$

We deduce that

$$\langle \phi, \mathbf{L}'\rangle = \lim_{n\to\infty} \langle \phi, \mathbf{A}'_n\rangle.$$

According to Lemma 27, it follows that the same equality holds for every $\phi \in \mathrm{FO}^{\mathrm{local}}(\lambda)$ hence $\mathbf{L}'$ is a modeling $\mathrm{FO}^{\mathrm{local}}$-limit of the sequence $(\mathbf{A}'_n)_{n\in\mathbb{N}}$.

As above, for $\lambda_a > 0$, if $\mathbf{L}'$ is a modeling $\mathrm{FO}^{\mathrm{local}}$-limit of $(\mathbf{A}'_n)_{n\in\mathbb{N}}$ then it is a modeling FO-limit.

□ □

Lemma 34. *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be an* FO*-convergent sequence of finite λ-structures, with component relation ϖ. Assume all the $\mathbf{A}_n$ have at most k connected components. Denote by $\mathbf{B}_{n,1}, \dots, \mathbf{B}_{n,k}$ these components (and additional empty λ-structures if necessary).*

Assume that for each $1 \le i \le k$ it holds $\lim_{n\to\infty} |B_{n,i}|/|A_n| = 1/k$.

Then there exists a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of permutations of $[k]$ such that for each $1 \le i \le k$ the sequence $(\mathbf{B}_{n,\sigma_n(i)})_{n\in\mathbb{N}}$ is FO*-convergent.*

Proof. To a formula $\phi \in \mathrm{FO}_p(\lambda)$ we associate the ϖ-local formula $\widetilde{\phi} \in \mathrm{FO}_p^{\mathrm{local}}(\lambda)$ asserting that all the free variables are ϖ-adjacent and that their closed neighborhood (that is their connected component) satisfies ϕ. Then essentially the same proof as above allows to refine $\mathbf{A}_n$ into sequences such that $\langle \phi, \mathbf{A}'_{n,i}\rangle$ is constant on the connected components of each of the $\mathbf{A}'_n$. Considering formulas allowing to split at least one of the sequences, we repeat this process (at most $k-1$ times) until each $\mathbf{A}'_{n,i}$ contains equivalent connected components. Then, $\mathbf{A}'_{n,i}$ can be split into connected components in an arbitrary order, thus obtaining the sequences $\mathbf{B}_{n,i}$. □ □

So we have proved that a FO-convergent can be decomposed by isolines of the ϖ-spectrum. In the next sections, we shall investigate how to refine further.

9.1. **Sequences with Finite Spectrum.** For every ϖ-nice sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ with finite support I, we define the *residue* $\mathbf{R}_n$ of $\mathbf{A}_n$ as the union of the connected components $\mathbf{B}_{n,i}$ of $\mathbf{A}_n$ such that $i \notin I$.

When one considers an $\mathrm{FO}^{\mathrm{local}}$-convergent sequence $(\mathbf{A}_n)$ with a finite support then the sequence of the residues forms a sequence which is either $\mathrm{FO}^{\mathrm{local}}$-convergent or "negligible" in the sense that $\lim_{n\to\infty} |R_n|/|A_n| = 0$. This is formulated as follows:

Lemma 35. *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a sequence of λ-structures with component relation ϖ. For each $n \in \mathbb{N}$ and $i \in \mathbb{N}$, let $\mathbf{B}_{n,i}$ be the i-th largest connected component of $\mathbf{A}_n$.*

Assume that $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent and has finite spectrum $(\lambda_i)_{i\in I}$. Let $\mathbf{R}_n$ be the residue of $\mathbf{A}_n$.

Then $\lambda' = \lim_{n\to\infty} |R_n|/|A_n|$ exists and either $\lambda' = 0$ or $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent.

Proof. Clearly, $\lambda' = 1 - \sum_i \lambda_i$. Assume $\lambda' > 0$. First notice that for every $\epsilon > 0$ there exists N such that for every $i > N$, the λ-structure $\mathbf{R}_n$ has no connected component of size at least $\epsilon/2\lambda'|A_n|$ and $\mathbf{R}_n$ has order at least $\lambda'/2|A_n|$. Hence, for every $i > N$, the λ-structure $\mathbf{R}_n$ has no connected component of size at least $\epsilon|R_n|$. According to Lemma 32, proving that $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent reduces to proving that $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent.

Let $\phi \in \mathrm{FO}_1^{\mathrm{local}}$. We group the λ-structures $\mathbf{B}_{n,i}$ (for $i \in I$) by values of λ_i as $\mathbf{A}'_{n,1}, \dots, \mathbf{A}'_{n,q}$. Denote by c_j the common value of λ_i for the connected components $\mathbf{B}_{n,i}$ in $\mathbf{A}'_{n,j}$. According to Corollary 2 it holds (as ϕ is clearly ϖ-local):

$$\begin{aligned}\langle \phi, \mathbf{A}_n\rangle &= \sum_i \frac{|B_{n,i}|}{|A_n|}\langle \phi, \mathbf{B}_{n,i}\rangle\\ &= \sum_{i\in I} \frac{|B_{n,i}|}{|A_n|}\langle \phi, \mathbf{B}_{n,i}\rangle + \sum_{i\notin I} \frac{|B_{n,i}|}{|A_n|}\langle \phi, \mathbf{B}_{n,i}\rangle\\ &= \sum_{j=1}^{q} \frac{|A'_{n,j}|}{|A_n|}\langle \phi, \mathbf{A}'_{n,j}\rangle + \frac{|R_n|}{|A_n|}\langle \phi, \mathbf{R}_n\rangle\end{aligned}$$

According to Lemma 33, each sequence $(\mathbf{A}'_{n,j})_{n\in\mathbb{N}}$ is FO-convergent. Hence the limit $\lim_{n\to\infty}\langle \phi, \mathbf{R}_n\rangle$ exists and we have

$$\lim_{n\to\infty}\langle \phi, \mathbf{R}_n\rangle = \frac{1}{\lambda'}\Bigl(\lim_{n\to\infty}\langle \phi, \mathbf{A}_n\rangle - \sum_{j=1}^{q} c_j \lim_{n\to\infty}\langle \phi, \mathbf{A}'_{n,j}\rangle\Bigr).$$

It follows that the sequence $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent. □ □

The following result finally determines the structure of converging sequences of (disconnected) λ-structures with finite support. This structure is called *comb structure*, see Fig 6.

Theorem 23 (Comb structure for λ-structure sequences with finite spectrum)**.** *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be an* $\mathrm{FO}^{\mathrm{local}}$*-convergent sequence of finite λ-structures with component relation ϖ and finite spectrum $(\lambda_i)_{i\in I}$. Let $\mathbf{R}_n$ be the residue of $\mathbf{A}_n$.*

Then there exists, for each $n \in \mathbb{N}$, a permutation $f_n : I \to I$ such that it holds

- $\lim_{n\to\infty} \max_{i\notin I} |B_{n,i}|/|A_n| = 0$*;*
- $\lim_{n\to\infty} |R_n|/|A_n|$ *exists;*
- *for every $i \in I$, the sequence $(B_{n,f_n(i)})_{n\in\mathbb{N}}$ is* FO*-convergent and* $\lim_{n\to\infty} |B_{n,f_n(i)}|/|A_n| = \lambda_i$*;*
- *either* $\lim_{n\to\infty} |R_n|/|A_n| = 0$*, or the sequence $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is* $\mathrm{FO}^{\mathrm{local}}$*-convergent.*

Moreover, if $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is FO*-convergent then $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is elementary-convergent.*

Proof. This lemma is a direct consequence of Lemmas 33, 34 and 35, except that we still have to prove FO-convergence of $(\mathbf{R}_n)_{n\in\mathbb{N}}$ in the case where $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is FO-convergent. As I is finite, the elementary convergence of $(\mathbf{R}_n)_{n\in\mathbb{N}}$ easily follows from the one of $(\mathbf{A}_n)$ and the one of the $(\mathbf{B}_{n,f_n(i)})$ for $i \in I$. □ □

9.2. **Sequences with Infinite Spectrum.** Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a ϖ-nice sequence with infinite spectrum (and support $I = \mathbb{N}$). In such a case, the notion of a residue becomes more tricky and will need some technical definitions. Before this, let us take the time to give an example illustrating the difficulty of the determination of the residue $\mathbf{R}_n$ in the comb structure of sequences with infinite spectrum.

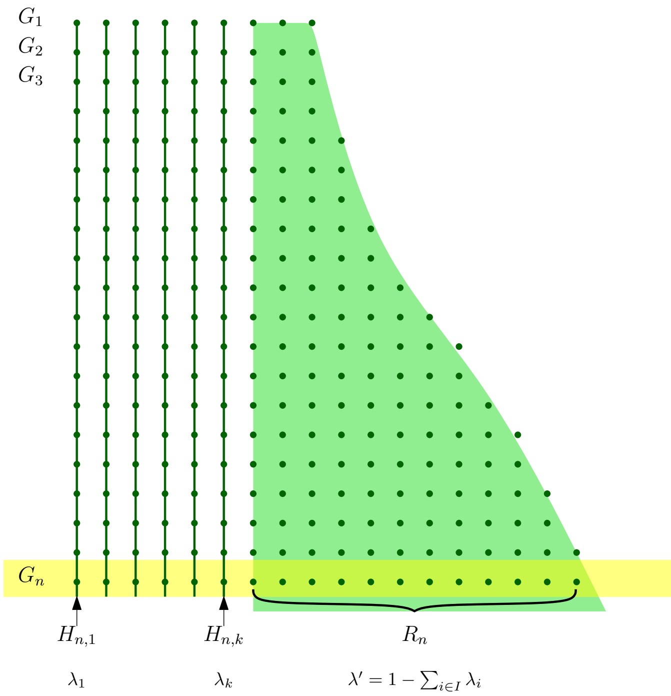

FIGURE 6. Illustration of the Comb structure for sequences with finite support

Example 9. Consider the sequence $(G_n)_{n\in\mathbb{N}}$ where G_n is the union of 2^n stars $H_{n,1},\dots,H_{n,2^n}$, where the i-th star $H_{n,i}$ has order $2^{2^n}(2^{-i}+2^{-n})/2$. Then it holds

$$\lambda_i = \lim_{n\to\infty} |H_{n,i}|/|G_n| = 2^{-(i+1)}$$

hence $\sum_i \lambda_i = 1/2$ thus the residue asymptotically should contain half of the vertices of G_n! An FO-limit of this sequence is shown Fig. 7.

This example is not isolated. In fact it is quite frequent in many of its variants. To decompose such examples we need a convenient separation. This is provided by the notion of clip.

Definition 17.
- A *clip* of a ϖ-nice sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ with support $\mathbb{N}$ is a non-decreasing function $C:\mathbb{N}\to\mathbb{N}$ such that $\lim_{n\to\infty} C(n)=\infty$ and

$$\forall n'\geq n\quad \sum_{i=1}^{C(n)}\left|\frac{|B_{n',i}|}{|A_{n'}|}-\lambda_i\right| \leq \sum_{i>C(n)}\lambda_i$$

- The *residue* $\mathbf{R}_n$ of $\mathbf{A}_n$ with respect to a clip $C(n)$ is the disjoint union of the $\mathbf{B}_{n,i}$ for $i>C(n)$.

Proposition 4. *Every ϖ-nice sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ with infinite support has a clip C_0, which is defined by*

$$C_0(n)=\sup\left\{M,\quad (\forall n'\geq n)\ \sum_{i=1}^{M}\left|\frac{|B_{n',i}|}{|A_{n'}|}-\lambda_i\right| \leq \sum_{i>M}\lambda_i\right\}.$$

Moreover, $\lim_{n\to} C_0(n)=\infty$ *and a non decreasing function C is a clip of* $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *if and only if* $C\leq C_0$ *and* $\lim_{n\to} C(n)=\infty$.

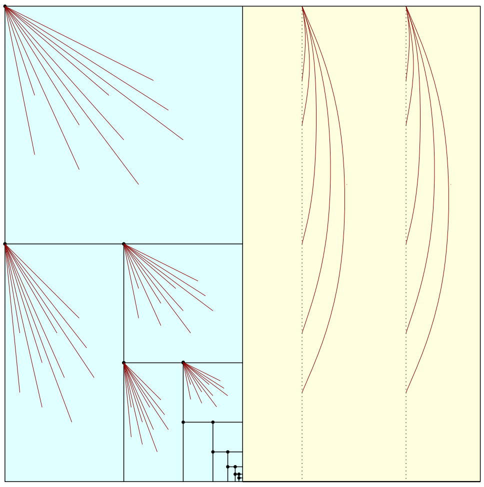

FIGURE 7. An FO-local limit. On the left side, each rectangle correspond to a star with the upper left point as its center; on the right side, each vertical line is a star with the upper point as its center.

Proof. Indeed, for each $n \in \mathbb{N}$, the value $z_l(M) = \sup_{n' \geq n} \sum_{i=1}^{M} \left| \frac{|B_{n',i}|}{|A_{n'}|} - \lambda_i \right|$ is non-decreasing function of C with $z_l(0) = 0$, and $z_r(M) = \sum_{i>M} \lambda_i$ is a decreasing function of C with $z_r(0) = \sum_i \lambda_i > 0$ hence C_0 is well defined. Moreover, for every integer M, let $\alpha = \sum_{i>M} \lambda_i > 0$. Then, as $\lim_{n\to\infty} |B_{n',i}|/|A_{n'}| = \lambda_i$ there exists N such that for every $n' \geq N$ and every $1 \leq i \leq M$ it holds $||B_{n',i}|/|A_{n'}| - \lambda_i| \leq \alpha/M$ thus for every $n' \geq N$ it holds

$$\sum_{i=1}^{M} \left| \frac{|B_{n',i}|}{|A_{n'}|} - \lambda_i \right| \leq \alpha = \sum_{i>M} \lambda_i.$$

It follows that $C_0(N) \geq M$. Hence $\lim_{n\to\infty} C_0(n) = \infty$.

That a non decreasing function C is a clip of $(\mathbf{A}_n)_{n\in\mathbb{N}}$ if and only if $C \leq C_0$ and $\lim_{n\to} C(n) = \infty$ follows directly from the definition. □ □

Lemma 36. *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a ϖ-nice sequence with support $\mathbb{N}$, and let C be a clip of $(\mathbf{A}_n)_{n\in\mathbb{N}}$.*

Then the limit $\lambda' = \lim_{n\to\infty} \frac{|R_n|}{|A_n|}$ exists and $\lambda' = 1 - \sum_i \lambda_i$.

Proof. As C is a clip, it holds for every $n \in \mathbb{N}$

$$\sum_i \lambda_i - 2 \sum_{i>C(n)} \lambda_i \leq \sum_{i=1}^{C(n)} \frac{|B_{n,i}|}{|A_n|} \leq \sum_i \lambda_i.$$

Also, for every $\epsilon > 0$ there exists n such that $|\sum_{i=1}^{C(n)} \lambda_i - \sum_i \lambda_i| < \epsilon$, that is: $\sum_{i>C(n)} \lambda_i < \epsilon$. It follows that

$$\lim_{n\to\infty} \sum_{i=1}^{C(n)} \frac{|B_{n,i}|}{|A_n|} = \sum_i \lambda_i.$$

Hence the limit $\lambda' = \lim_{n\to\infty} \frac{|R_n|}{|A_n|}$ exists and $\lambda' = 1 - \sum_i \lambda_i$. □ □

Lemma 37. *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a sequence of λ-structures with component relation ϖ. For each $n\in\mathbb{N}$ and $i\in\mathbb{N}$, let $\mathbf{B}_{n,i}$ be the i-th largest connected component of $\mathbf{A}_n$ (if i is at most equal to the number of connected components of $\mathbf{A}_n$, the empty λ-structure otherwise).*

Assume that $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is FO*-convergent.*

Let $C:\mathbb{N}\to\mathbb{N}$ be a clip of $(\mathbf{A}_n)_{n\in\mathbb{N}}$, and let $\mathbf{R}_n$ be the residue of $\mathbf{A}_n$ with respect to C.

Let $\lambda'=\lim_{n\to\infty}|R_n|/|A_n|$. Then either $\lambda'=0$ or $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$*-convergent.*

Proof. According to Lemma 36, $\lim_{n\to\infty}|R_n|/|A_n|$ exists and $\lambda'=1-\sum_i\lambda_i$. Assume $\lambda'>0$. First notice that for every $\epsilon>0$ there exists N such that for every $i>N$, the λ-structure $\mathbf{R}_n$ has no connected component of size at least $\epsilon/2\lambda'|\mathbf{A}_n|$ and $\mathbf{R}_n$ has order at least $\lambda'/2|A_n|$. Hence, for every $i>N$, the λ-structure $\mathbf{R}_n$ has no connected component of size at least $\epsilon|R_n|$. According to Lemma 32, proving that $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent reduces to proving that $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent.

Let $\phi\in\mathrm{FO}_1^{\mathrm{local}}$ (thus ϕ is ϖ-local). Let $\epsilon>0$. There exists $k\in\mathbb{N}$ such that $\sum_{i\leq k}\lambda_i>1-\lambda'-\epsilon/3$ and such that $\lambda_{k+1}<\lambda_k$. We group the λ-structures $\mathbf{B}_{n,i}$ (for $1\leq i\leq k$) by values of λ_i as $\mathbf{A}'_{n,1},\dots,\mathbf{A}'_{n,q}$. Denote by c_j the common value of λ_i for the connected components $\mathbf{B}_{n,i}$ in $\mathbf{A}'_{n,j}$. According to Lemma 33, each sequence $(\mathbf{A}'_{n,i})_{n\in\mathbb{N}}$ is FO-convergent. Define

$$\mu_i=\lim_{n\to\infty}\langle\phi,\mathbf{A}'_{n,i}\rangle.$$

There exists N such that for every $n>N$ it holds

$$\sum_{i=1}^{q}|\langle\phi,\mathbf{A}'_{n,i}\rangle-\mu_i|<\epsilon/3.$$

According to Corollary 2 it holds, for every $n>N$:

$$\begin{aligned}\langle\phi,\mathbf{A}_n\rangle&=\sum_i\frac{|B_{n,i}|}{|A_n|}\langle\phi,\mathbf{B}_{n,i}\rangle\\&=\sum_{i=1}^{k}\frac{|B_{n,i}|}{|A_n|}\langle\phi,\mathbf{B}_{n,i}\rangle+\sum_{i=k+1}^{C(n)}\frac{|A_{n,i}|}{|A_n|}\langle\phi,\mathbf{B}_{n,i}\rangle+\sum_{i>C(n)}\frac{|B_{n,i}|}{|A_n|}\langle\phi,\mathbf{B}_{n,i}\rangle\\&=\sum_{i=1}^{q}c_i\langle\phi,\mathbf{A}'_{n,i}\rangle+\sum_{i=k+1}^{C(n)}\frac{|B_{n,i}|}{|A_n|}\langle\phi,\mathbf{B}_{n,i}\rangle+\frac{|R_n|}{|A_n|}\langle\phi,\mathbf{R}_n\rangle\end{aligned}$$

Thus we have

$$\begin{aligned}\Big|\lambda'\langle\phi,\mathbf{R}_n\rangle-\big(\langle\phi,\mathbf{A}_n\rangle-\sum_{i=1}^{q}c_i\mu_i\big)\Big|&\leq\sum_{i=1}^{q}|\langle\phi,\mathbf{A}'_{n,i}\rangle-\mu_i|+\sum_{i=k+1}^{C(n)}|B_{n,i}|/|A_n|+\big||R_n|/|A_n|-\lambda'\big|\\&\leq\epsilon.\end{aligned}$$

It follows that $\lim_{n\to\infty}\langle\phi,\mathbf{R}_n\rangle$ exists. By sorting the $C(n)$ first connected components of each $\mathbf{A}_n$ according to both λ_i and Lemma 34 we obtain the following expression for the limit:

$$\lim_{n\to\infty}\langle\phi,\mathbf{R}_n\rangle=\frac{1}{\lambda'}\big(\lim_{n\to\infty}\langle\phi,\mathbf{A}_n\rangle-\sum_{i<\widehat{C}}\lambda_i\lim_{n\to\infty}\langle\phi,\mathbf{B}_{n,i}\rangle\big).$$

□ □

Theorem 24 (Comb structure for λ-structure sequences with infinite spectrum (local convergence))**.** *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be an $\mathrm{FO}^{\mathrm{local}}$-convergent sequence of finite λ-structures with component relation ϖ, support $\mathbb{N}$, and spectrum $(\lambda_i)_{i\in\mathbb{N}}$. Let $C:\mathbb{N}\to\mathbb{N}$ be a clip of $(\mathbf{A}_n)_{n\in\mathbb{N}}$, and let $\mathbf{R}_n$ be the residue of $\mathbf{A}_n$ with respect to C.*

Then there exists, for each $n\in\mathbb{N}$, a permutation $f_n:[C(n)]\to[C(n)]$ such that, extending f_n to $\mathbb{N}$ by putting $f(i)$ to be the identity for $i>C(n)$, it holds

- $\lim_{n\to\infty}\max_{i>C(n)}|B_{n,i}|/|A_n|=0$;
- $\lambda'=\lim_{n\to\infty}|R_n|/|A_n|$ *exists;*
- *for every $i\in\mathbb{N}$, $(B_{n,f_n(i)})_{n\in\mathbb{N}}$ is* FO*-convergent;*
- *either $\lambda'=0$ or the sequence $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent.*

Proof. This lemma is a direct consequence of the previous lemmas. □ □

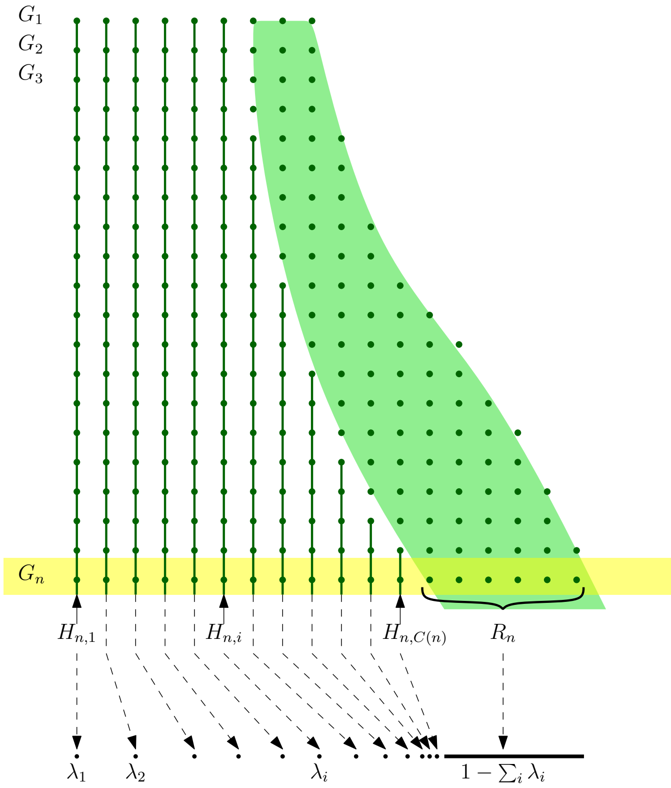

FIGURE 8. Illustration of the Comb structure theorem

We shall now extend the Comb structure theorem to full FO-convergence. Opposite to the case of a finite ϖ-spectrum, the elementary convergence aspects will be non trivial and will require a careful choice of a clip for the sequence.

Lemma 38. *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be an $\mathrm{FO}^{\mathrm{local}}$-convergent sequence of finite λ-structures with component relation ϖ, such that $\lim_{n\to\infty}|A_n|=\infty$. Let $\mathbf{B}_{n,i}$ be the connected components of $\mathbf{A}_n$. Assume that the connected components with same λ_i have been reshuffled according to Lemma 34, so that $(\mathbf{B}_{n,i})_{i\in\mathbb{N}}$ is* FO*-convergent for each $i\in\mathbb{N}$.*

For $i\in\mathbb{N}$, let $\widehat{\mathbf{B}_i}$ be an elementary limit of $(\mathbf{B}_{n,i})_{n\in\mathbb{N}}$. Then there exists a clip C such that the sequence $(\mathbf{R}_n)_{n\in\mathbb{N}}$ of the residues is elementarily convergent.

Moreover, if $\widehat{\mathbf{R}}$ *is an elementary limit of* $(\mathbf{R}_n)_{n\in\mathbb{N}}$*, then* $\bigcup_i \widehat{\mathbf{B}_i} \cup \widehat{\mathbf{R}}$ *is an elementary limit of* $(\mathbf{A}_n)_{n\in\mathbb{N}}$.

Let $\mathbf{B}'_{n,i}$ *be either* $\mathbf{B}_{n,i}$ *if* $C(n) \geq i$ *or the empty* λ*-structure if* $C(n) < i$*. Then the family consisting in the sequences* $(\mathbf{B}'_{n,i})_{i\in\mathbb{N}}$ *(*$i \in \mathbb{N}$*) and of the sequence* $(\mathbf{R}_n)_{n\in\mathbb{N}}$ *is uniformly elementarily convergent.*

Proof. Let $\widehat{\mathbf{A}}$ be an elementary limit of $(\mathbf{A}_n)_{n\in\mathbb{N}}$.

For $\theta \in \mathrm{FO}_1^{\mathrm{local}}$ and $m \in \mathbb{N}$ we denote by $\theta^{(m)}$ the sentence

$$\theta^{(m)}: \quad (\exists x_1 \dots \exists x_m) \left(\bigwedge_{1\leq i<j\leq m} \neg\varpi(x_i, x_j) \ \wedge \ \bigwedge_{i=1}^{m} \theta(x_i) \right).$$

According to Theorem 10, elementary convergence of a sequence of λ-structures with component relation ϖ can be checked by considering sentences of the form $\theta^{(k)}$ for $\theta \in \mathrm{FO}_1^{\mathrm{local}}$ and $k \in \mathbb{N}$.

Note that for every $k < k'$ and every λ-structure $\mathbf{A}$, if it holds $\mathbf{A} \models \theta^{(k')}$ then it holds $\mathbf{A} \models \theta^{(k)}$. Define

$$M(\theta) = \sup\{k \in \mathbb{N}, \quad \widehat{\mathbf{A}} \models \theta^{(k)}\}$$
$$\Omega(\theta) = \{i \in \mathbb{N}, \quad \widehat{\mathbf{B}_i} \models (\exists x)\theta(x)\}.$$

Note that obviously $|\Omega(\theta)| \leq M(\theta)$.

For $r \in \mathbb{N}$, let $\theta_1, \dots, \theta_{F(r)}$ be an enumeration of the local first-order formulas with a single free variable with quantifier rank at most r (up to logical equivalence). Define

$$A(r) = \max(r, \max_{a\leq F(r)} \max \Omega(\theta_a)).$$

Let

$$C_0(n) = \sup\left\{K, \quad (\forall n' \geq n) \ \sum_{i=1}^{K} \left| \frac{|B_{n',i}|}{|A_{n'}|} - \lambda_i \right| \leq \sum_{i>K} \lambda_i \right\}$$

be the standard (maximal) clip on $(\mathbf{A}_n)_{n\in\mathbb{N}}$ (see Proposition 4).

Let $B(r)$ be the minimum integer such that

(1) it holds $C_0(B(r)) \geq A(r)$ (note that $\lim_{n\to\infty} C_0(n) = \infty$, according to Proposition 4);
(2) for every $n \geq B(r), a \leq F(r)$ and every $k \leq r$ it holds $\mathbf{A}_n \models \theta_a^{(k)}$ if and only if $M(\theta_a) \geq k$ (note that this holds for sufficiently large n as $\widehat{\mathbf{A}}$ is an elementary limit of $(\mathbf{A}_n)_{n\in\mathbb{N}}$);
(3) for every $i \leq A(r)$ and $a \leq F(r)$ it holds

$$\mathbf{B}_{n,i} \models (\exists x)\theta_a(x) \quad \iff \quad \widehat{\mathbf{B}_i} \models (\exists x)\theta_a(x).$$

(note that this holds for sufficiently large n as $\widehat{\mathbf{B}_i}$ is an elementary limit of $(\mathbf{B}_{n,i})_{n\in\mathbb{N}}$ and as we consider only finitely many values of i);

we define the non-decreasing function $C : \mathbb{N} \to \mathbb{N}$ by

$$C(n) = \max\{A(r) : B(r) \leq n\}.$$

As $\lim_{r\to\infty} A(r) = \infty$ and as $C_0(B(r)) \geq A(r)$ it holds $\lim_{r\to\infty} B(r) = \infty$. Moreover, for every $r \in \mathbb{N}$ it holds $C_0(B(r)) \geq A(r)$ hence $C_0(n) \geq C(n)$. According to Proposition 4, it follows that the function C is a clip on $(\mathbf{A}_n)_{n\in\mathbb{N}}$.

Let $(\mathbf{R}_n)_{n\in\mathbb{N}}$ be the resiude of $(\mathbf{A}_n)_{n\in\mathbb{N}}$ with respect to the clip C, and let $\mathbf{B}'_{n,i}$ be defined as $\mathbf{B}_{n,i}$ if $i \leq C(n)$ and the empty λ-structure otherwise. Then it is immediate from the definition of the clip C that the family $\{(B'_{n,i})_{n\in\mathbb{N}} : \ i \in \mathbb{N}\}$ is uniformly elementarily convergent. Using Lemma 28, it is also easily checked that the residue $(\mathbf{R}_n)_{n\in\mathbb{N}}$ of $(\mathbf{A}_n)_{n\in\mathbb{N}}$ with respect to the clip C is elementarily

convergent and thus, that the family $\{(B'_{n,i})_{n\in\mathbb{N}} :\ i \in \mathbb{N}\} \cup \{(\mathbf{R}_n)_{n\in\mathbb{N}}\}$ is uniformly elementarily convergent. □ □

The extension of the Comb structure theorem now follows directly.

Theorem 25 (Comb structure for λ-structure sequences with infinite spectrum)**.** *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be an* FO*-convergent sequence of finite λ-structures with component relation ϖ and infinite spectrum $(\lambda_i)_{i\in\mathbb{N}}$.*

Then there exists a clip $C : \mathbb{N} \to \mathbb{N}$ with residue $\mathbf{R}_n$ and, for each $n \in \mathbb{N}$, a permutation $f_n : [C(n)] \to [C(n)]$ such that, extending f_n to $\mathbb{N}$ by putting $f(i)$ to be the identity for $i > C(n)$, and letting $\mathbf{B}'_{n,i}$ be either $\mathbf{B}_{n,f_n(i)}$ if $C(n) \geq i$ or the empty λ-structure if $C(n) < i$, it holds:

- $\mathbf{A}_n = \mathbf{R}_n \cup \bigcup_{i\in\mathbb{N}} \mathbf{B}'_{n,i}$*;*
- $\lim_{n\to\infty} \max_{i>C(n)} |B'_{n,i}|/|A_n| = 0$*;*
- $\lim_{n\to\infty} |R_n|/|A_n|$ *exists;*
- *for every $i \in \mathbb{N}$, $(\mathbf{B}'_{n,i})_{n\in\mathbb{N}}$ is* FO*-convergent;*
- *either $\lim_{n\to\infty} |R_n|/|A_n| = 0$ and $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is elementarily convergent, or the sequence $(\mathbf{R}_n)_{n\in\mathbb{N}}$ is* FO*-convergent;*
- *the family $\{(\mathbf{B}'_{n,i})_{n\in\mathbb{N}} : i \in \mathbb{N}\} \cup \{(\mathbf{R}_n)_{n\in\mathbb{N}}\}$ is uniformly elementarily convergent.*

□

10. Limit of Colored Rooted Trees with Bounded Height

In this section we explicitly define modeling FO-limits for FO-convergent sequences of colored rooted trees with bounded height.

For the sake of simplicity, we first sketch our method for FO-convergent sequences of colored rooted trees with bounded height.

We consider two signatures:

(1) the signature λ consisting in a binary relation $\sim$ (adjacency), a unary relation R (property of being a root), and c unary relations C_i (the coloring). Colored rooted trees will be encoded as λ-structures, and the class of colored rooted trees with height at most h will be denoted by $\mathcal{Y}^{(h)}$.

(2) the signature λ^+, which is the signature λ augmented by a new unary relation P. The signature λ^+ is used to encode colored rooted forests with a *principal* connected component, whose root will be marked by relation P instead of R. The class of colored rooted forests with a principal connected component and height at most h will be denoted by $\mathcal{F}^{(h)}$.

Classes $\mathcal{Y}^{(h)}$ and $\mathcal{F}^{(h)}$ obviously form basic elementary classes. Thus they can be axiomatized by a single axioms. These axioms are denoted $\eta_Y^{(h)}$ and $\eta_F^{(h)}$. For integer $p \geq 0$, we further introduce a short notation for the Stone spaces associated to the Lindenbaum-Tarski algebras of formulas on $\mathcal{Y}^{(h)}$ and $\mathcal{F}^{(h)}$ with p free variables included in $\{x_1, \dots, x_p\}$:

$$\mathfrak{Y}_p^{(h)} = S(\mathcal{B}(\mathrm{FO}_p(\lambda), \eta_Y^{(h)}))$$
$$\mathfrak{F}_p^{(h)} = S(\mathcal{B}(\mathrm{FO}_p(\lambda^+), \eta_F^{(h)})).$$

We consider three basic interpretation schemes:

(1) $\mathsf{I}_{Y\to F}$ is a basic interpretation scheme of λ^+-structures in λ-structures, which maps a colored rooted tree with height h into a colored rooted forest with height $h-1$ (the trees rooted at the sons of the former root) and a single vertex rooted principal component (the former root);

(2) $\mathsf{I}_{F\to Y}$ is a basic interpretation scheme of λ-structures in λ^+-structures, which maps a colored rooted forest with height h into a colored rooted tree by making each non principal root a son of the principal root;
(3) $\mathsf{I}_{R\to P}$ is a basic interpretation scheme of λ-structures in λ^+-structures, which maps a colored rooted tree into the "identical" colored rooted forest having a single component. (Roughly speaking, the relation R becomes the relation P.)

Let $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ be an FO-convergent sequence of finite rooted colored trees such that $\lim_{n\to\infty}|Y_n| = \infty$.

According to the Comb Structure Theorem, there exists a countable set $(\mathbf{Y}_{n,i})_{n\in\mathbb{N}}$ of FO-convergent sequences of colored rooted trees and an FO-convergent sequence $(\mathbf{R}_n)_{n\in\mathbb{N}}$ of special colored rooted forests (as the isolated principal root obviously belongs to $\mathbf{R}_n$) forming a uniformly convergent family of sequences, such that $\mathsf{I}_{Y\to F}(\mathbf{Y}_n) = \mathbf{R}_n \cup \bigcup_{i\in I}\mathbf{Y}_{n,i}$.

If the limit spectrum of $(\mathsf{I}_{Y\to F}(\mathbf{Y}_n))_{n\in\mathbb{N}}$ is empty (i.e. $I=\emptyset$), the sequence $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ of colored rooted trees is called *residual*, and we prove directly that a residual sequence of colored rooted trees admit a modeling FO-limit, which is obtained by equipping a connected component of a universal relational sample space $\mathbb{Y}_h$ with a suitable probability measure.

Otherwise, we proceed by induction over the height bound h. Denote by $(\lambda_i)_{i\in I}$ the limit spectrum of $(I_{Y\to F}(\mathbf{Y}_n))_{n\in\mathbb{N}}$, let $\lambda_0 = 1-\sum_{i\in I}\lambda_i$, and let $\mathbf{Y}_{n,0} = \mathsf{I}_{R\to P}\circ\mathsf{I}_{F\to Y}(\mathbf{R}_n)$. As $(\mathsf{I}_{F\to Y}(\mathbf{R}_n))_{n\in\mathbb{N}}$ is residual, $(\mathbf{Y}_{n,0})_{n\in\mathbb{N}}$ has a modeling FO-limit $\widetilde{\mathbf{Y}}_0$. By induction, each $(\mathbf{Y}_{n,i})_{n\in\mathbb{N}}$ has a modeling FO-limit $\widetilde{\mathbf{Y}}_i$. As $\mathbf{Y}_n = \mathsf{I}_{F\to Y}(\bigcup_{i\in I\cup\{0\}}\mathbf{Y}_{n,i})$, we deduce (using uniform elementary convergence) that $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ has modeling FO-limit $\mathsf{I}_{F\to Y}(\coprod_{i\in I\cup\{0\}}(\widetilde{\mathbf{Y}}_i,\lambda_i))$.

This finishes the outline of our construction. Now we provide details.

10.1. **Preliminary Observations.** We take some time for some preliminary observations on the logic structure of rooted colored trees with bounded height, that will be of great help in our developments. These observations will be an occasion to see arguments based on Ehrenfeucht-Fraïssé games and strategy stealing. (Definitions of $\equiv^n$ and Ehrenfeucht-Fraïssé games were recalled in Section 5.1.)

For a rooted colored tree $\mathbf{Y}$ and a vertex $x\in Y$, we denote $\mathbf{Y}(x)$ the subtree of $\mathbf{Y}$ rooted at x, and by $\mathbf{Y}\setminus\mathbf{Y}(x)$ the rooted tree obtained from $\mathbf{Y}$ by removing all the vertices in $\mathbf{Y}(s)$.

Lemma 39. *Let $\mathbf{Y},\mathbf{Y}'$ be colored rooted trees with height h, let s,s' be sons of the roots of $\mathbf{Y}$ and $\mathbf{Y}'$, respectively.*

Let $n\in\mathbb{N}$. If $\mathbf{Y}(s)\equiv^n\mathbf{Y}'(s')$ and $\mathbf{Y}\setminus\mathbf{Y}(s)\equiv^n\mathbf{Y}'\setminus\mathbf{Y}'(s')$, then $\mathbf{Y}\equiv^n\mathbf{Y}'$.

Proof. Assume $\mathbf{Y}(s)\equiv^n\mathbf{Y}'(s')$ and $\mathbf{Y}\setminus\mathbf{Y}(s)\equiv^n\mathbf{Y}'\setminus\mathbf{Y}'(s')$. In order to prove $\mathbf{Y}\equiv^n\mathbf{Y}'$ we play an n-steps Ehrenfeucht-Fraïssé-game EF_0 on $\mathbf{Y}$ and $\mathbf{Y}'$ as Duplicator. Our strategy will be based on two auxiliary n-steps Ehrenfeucht-Fraïssé-games, EF_1 and EF_2, on $\mathbf{Y}(s)$ and $\mathbf{Y}'(s')$ and on $\mathbf{Y}\setminus\mathbf{Y}(s)$ and $\mathbf{Y}'\setminus\mathbf{Y}'(s')$, respectively, against Duplicators following a winning strategy. Each time Spoiler selects a vertex in game EF_0, we play the same vertex in the game EF_1 or EF_2 (depending on the tree the vertex belongs to), then we mimic the selection of the Duplicator of this game. it is easily checked that this strategy is a winning strategy. □ □

Lemma 40. *Let $\mathbf{Y},\mathbf{Y}'$ be colored rooted trees with height h, let s,s' be sons of the roots of $\mathbf{Y}$ and $\mathbf{Y}'$, respectively.*

Let $n\in\mathbb{N}$. If $\mathbf{Y}\equiv^{n+h}\mathbf{Y}'$ and $\mathbf{Y}(s)\equiv^n\mathbf{Y}'(s')$, then $\mathbf{Y}\setminus\mathbf{Y}(s)\equiv^n\mathbf{Y}'\setminus\mathbf{Y}'(s')$

Proof. Assume $\mathbf{Y}\equiv^{n+h}\mathbf{Y}'$ and $\mathbf{Y}(s)\equiv^n\mathbf{Y}'(s')$.

We first play (as Spoiler) s in $\mathbf{Y}$ then s' in $\mathbf{Y}'$. Let t' and t be the corresponding plays of Duplicator. Then the further n steps of the game have to map vertices in $\mathbf{Y}(s)$, $\mathbf{Y}(t)$, $\mathbf{Y}\setminus(\mathbf{Y}(s)\cup\mathbf{Y}(t))$ to $\mathbf{Y}'(t')$, $\mathbf{Y}'(s')$, $\mathbf{Y}'\setminus(\mathbf{Y}'(t')\cup\mathbf{Y}'(s'))$ (and converse), for otherwise $h-2$ steps would allow Spoiler to win the game. Also, by restricting our play to one of these pairs of trees, we deduce $\mathbf{Y}(s)\equiv^n \mathbf{Y}'(t')$, $\mathbf{Y}(t)\equiv^n \mathbf{Y}'(s')$, and $\mathbf{Y}\setminus(\mathbf{Y}(s)\cup\mathbf{Y}(t))\equiv^n \mathbf{Y}\setminus(\mathbf{Y}'(s')\cup\mathbf{Y}'(t'))$. As $\mathbf{Y}'(s')\equiv^n \mathbf{Y}(s)$ it follows

$$\mathbf{Y}(t)\equiv^n \mathbf{Y}'(s')\equiv^n \mathbf{Y}(s)\equiv^n \mathbf{Y}'(t').$$

Hence, according to Lemma 39, as $\mathbf{Y}\setminus(\mathbf{Y}(s)\cup\mathbf{Y}(t))=(\mathbf{Y}\setminus\mathbf{Y}(s))\setminus\mathbf{Y}(t)$ and $\mathbf{Y}'\setminus(\mathbf{Y}'(s')\cup\mathbf{Y}'(t'))=(\mathbf{Y}'\setminus\mathbf{Y}'(s'))\setminus\mathbf{Y}'(t')$, we deduce $\mathbf{Y}\setminus\mathbf{Y}(s)\equiv^n \mathbf{Y}'\setminus\mathbf{Y}'(s')$. □ □

Lemma 41. *Let λ be a signature of colored rooted tree and let λ^+ be the signature λ augmented by a unary symbol M (interpreted as a marking).*

Let $\mathbf{Y},\mathbf{Y}'$ be colored rooted trees with height at most h with signature λ^+, such that $\mathbf{Y}$ (resp. $\mathbf{Y}'$) has exactly one marked vertex m (resp. m'). Assume that both m and m' have height $t>1$ (in $\mathbf{Y}$ and $\mathbf{Y}'$, respectively). Let s (resp. s') be the ancestor of m (resp. m') at height 2. Let Unmark *be the interpretation of λ-structures in λ^+-structures consisting in forgetting M and let $n\in\mathbb{N}$.*

If $\mathrm{Unmark}(\mathbf{Y})\equiv^{n+h}\mathrm{Unmark}(\mathbf{Y}')$ *and* $\mathbf{Y}(s)\equiv^n\mathbf{Y}'(s')$, *then* $\mathbf{Y}\equiv^n\mathbf{Y}'$.

Proof. Assume $\mathrm{Unmark}(\mathbf{Y})\equiv^{n+h}\mathrm{Unmark}(\mathbf{Y}')$ and $\mathbf{Y}(s)\equiv^n\mathbf{Y}'(s')$. Then it holds $\mathrm{Unmark}(\mathbf{Y}(s))\equiv^n\mathrm{Unmark}(\mathbf{Y}'(s'))$ thus, according to Lemma 40,

$$\begin{aligned}\mathbf{Y}\setminus\mathbf{Y}(s)&=\mathrm{Unmark}(\mathbf{Y})\setminus\mathrm{Unmark}(\mathbf{Y}(s))\\&\equiv^n\mathrm{Unmark}(\mathbf{Y}')\setminus\mathrm{Unmark}(\mathbf{Y}'(s'))=\mathbf{Y}'\setminus\mathbf{Y}'(s').\end{aligned}$$

Hence, according to Lemma 39, it holds $\mathbf{Y}\equiv^n\mathbf{Y}'$. □ □

The next lemma states that the properties of a colored rooted trees with a distinguished vertex v can be retrieved from the properties of the subtree rooted at v, the subtree rooted at the father of v, etc. (see Fig. 9).

Lemma 42. *Let $\mathbf{Y},\mathbf{Y}'$ be colored rooted trees with height at most h, $v_t\in Y$ and $v_t'\in Y'$ be vertices with height t. For $1\le i<t$, let v_i (resp. v_i') be the ancestor of v_t (resp. of v_t') at height i.*

Then for every integer n it holds

$$(\forall 1\le i\le t)\ \mathbf{Y}(v_i)\equiv^{n+h+1-i}\mathbf{Y}'(v_i')\implies(\mathbf{Y},v_t)\equiv^n(\mathbf{Y}',v_t')$$
$$(\mathbf{Y},v_t)\equiv^{n+(t-1)h}(\mathbf{Y}',v_t')\implies(\forall 1\le i\le t)\ \mathbf{Y}(v_i)\equiv^{n+(t-i)h}\mathbf{Y}'(v_i')$$

Proof. We proceed by induction over t. If $t=1$, then the statement obviously holds. So, assume $t>1$ and that the statement holds for $t-1$.

Let λ be the signature of $\mathbf{Y}$ and $\mathbf{Y}'$, let λ^+ be the signature obtained by adding to λ a unary symbol M (interpreted as marking), and let $\mathbf{Y}_+$ (resp. $\mathbf{Y}_+'$) be the rooted colored trees (with signature λ^+) obtained from $\mathbf{Y}$ (resp. $\mathbf{Y}'$) by marking v_t (resp. v_t').

Assume $(\forall 1\le i\le t)\ \mathbf{Y}(v_i)\equiv^{n+h+1-i}\mathbf{Y}'(v_i')$. By induction, $(\forall 2\le i\le t)\ \mathbf{Y}(v_i)\equiv^{n+(h-1)+1-(i-1)}\mathbf{Y}'(v_i')$ implies $(\mathbf{Y}(v_2),v_t)\equiv^n(\mathbf{Y}'(v_2'),v_t')$, that is $\mathbf{Y}_+(v_2)\equiv^n\mathbf{Y}_+'(v_2')$. As $\mathbf{Y}\equiv^{n+h}\mathbf{Y}'$, it follows from Lemma 41 that $\mathbf{Y}_+\equiv^n\mathbf{Y}_+'$, that is: $(\mathbf{Y},v_t)\equiv^n(\mathbf{Y}',v_t')$.

Conversely, if $(\mathbf{Y},v_t)\equiv^{n+(t-1)h}(\mathbf{Y}',v_t')$ (i.e. $\mathbf{Y}_+\equiv^{n+(t-1)h}\mathbf{Y}_+'$) an repeated application of Lemma 40 gives $\mathbf{Y}_+(v_i)\equiv^{n+(t-i)h}\mathbf{Y}_+'(v_i)'$ hence (by forgetting the marking) $\mathbf{Y}(v_i)\equiv^{n+(t-i)h}\mathbf{Y}'(v_i)'$. □ □

This lemma allows to encode the complete theory of a colored rooted tree $\mathbf{Y}$ with height at most h and a single special vertex v as a tuple of complete theories of colored rooted trees with height at most h.

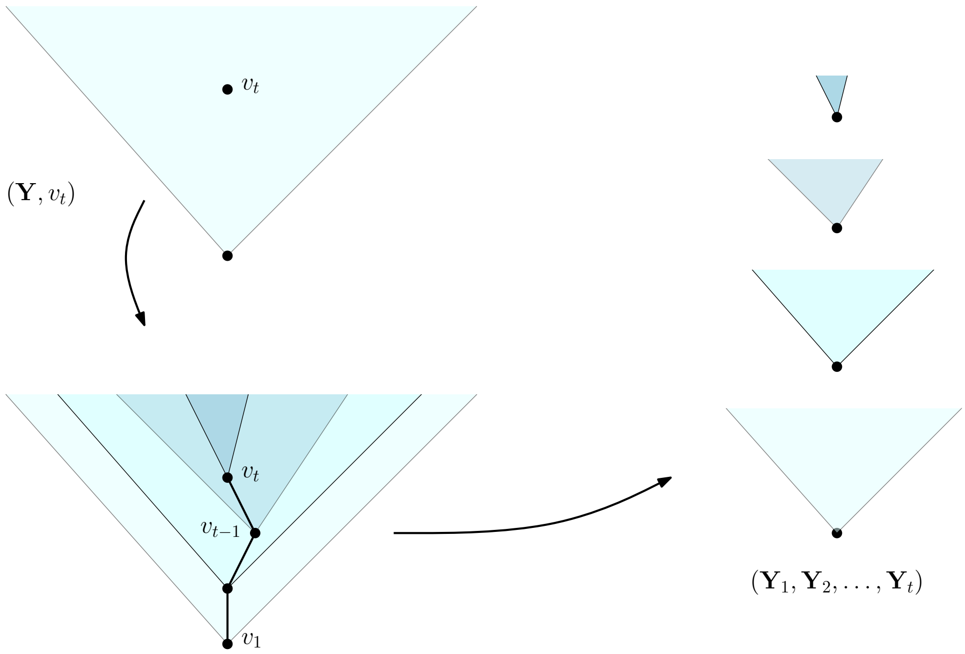

Figure 9. Transformation of a rooted tree with a distinguished vertex $(\mathbf{Y}, v_t)$ into a tuple of rooted trees $(\mathbf{Y}_1, \ldots, \mathbf{Y}_t)$.

10.2. **The universal relational sample space $\mathbb{Y}_h$.** The aim of this section is to construct a rooted colored forest on a standard Borel space $\mathbb{Y}_h$ that is *residual-universal*, in the sense that every residual sequence of colored rooted trees will have a modeling FO-limit obtained by assigning an adapted probability measure to one of the connected components of $\mathbb{Y}_h$.

For theories $T, T' \in \mathfrak{Y}_0^{(h)}$ (recall that $\mathfrak{Y}_0^{(h)} = S(\mathcal{B}(\mathrm{FO}_0(\lambda), \eta_Y^{(h)})))$, we define $w(T, T') \geq k$ if and only if there exists a model $\mathbf{Y}$ of T, such that the root of $\mathbf{Y}$ has k (distinct) sons $v_1, \ldots, v_k$ with $\mathrm{Th}(\mathbf{Y}(v_i)) = T'$.

For $z = (z_1, \ldots, z_a) \in \overline{\mathbb{N}}^a$ define the subset F_z of $(\mathfrak{Y}_0^{(h)})^{a+1}$ by

$$F_z = \{(T_0, \ldots, T_a) : w(T_{i-1}, T_i) = z_i\}.$$

For $t \in \overline{\mathbb{N}}$, define

$$X_t = \begin{cases} \{1, \ldots, t\}, & \text{if } t \in \mathbb{N}, \\ [0,1], & \text{otherwise} \end{cases}$$

For $z = (z_1, \ldots, z_a) \in \overline{\mathbb{N}}^a$, define $X_z = \prod_{i=1}^a X_{z_i}$. Let

$$V_h = \mathfrak{Y}_0^{(h)} \uplus \biguplus_z (F_z \times X_z).$$

Note that V_h is a subset of $\biguplus_{i=1}^h (\mathfrak{Y}_0^{(h)})^i \times [0,1]^{i-1}$. We define the σ-algebra Σ_h as the trace on V_h of the Borel σ-algebra of $\biguplus_{i=1}^h (\mathfrak{Y}_0^{(c,h)})^i \times [0,1]^{i-1}$.

Definition 18. The *universal forest* $\mathbb{Y}_h$ has vertex set V_h, set of roots $\mathfrak{Y}_0^{(h)}$, and edges

$$\{((T_0, T_1, \ldots, T_a), (\alpha_1, \ldots, \alpha_a)), ((T_0', T_1', \ldots, T_b'), (\alpha_1', \ldots, \alpha_b'))\}$$

where $|a - b| = 1$, $T_0 = T_0'$, and for every $1 \leq i \leq \min(a, b)$ it holds $T_i = T_i'$ and $\alpha_i = \alpha_i'$.

The remaining of this section will be devoted to the proof of Lemma 44, which states that $\mathbb{Y}_h$ is a relational sample space. In order to prove this result, we shall need a preliminary lemma, which expresses that the property of a tuple of vertices in a colored rooted tree with bounded height is completely determined by the individual properties of the vertices in the tuple and the heights of the lowest common ancestors of every pair of vertices in the tuples.

Lemma 43. *Fix rooted trees $\mathbf{Y}, \mathbf{Y}' \in \mathcal{Y}^{(h)}$. Let $u_1, \dots, u_p$ be p vertices of $\mathbf{Y}$, let $u'_1, \dots, u'_p$ be p vertices of $\mathbf{Y}'$, and let $n \in \mathbb{N}$.*

Assume that for every $1 \leq i \leq p$ it holds $(\mathbf{Y}, u_i) \equiv^{n+h} (\mathbf{Y}', u'_i)$ and that for every $1 \leq i, j \leq p$ the height of $u_i \wedge u_j$ in $\mathbf{Y}$ is the same as the height of $u'_i \wedge u'_j$ in $\mathbf{Y}'$ (where $u \wedge v$ denotes the lowest common ancestor of u and v).

Then $(\mathbf{Y}, u_i, \dots, u_p) \equiv^n (\mathbf{Y}', u'_1, \dots, u'_p)$.

Proof. In the proof we consider $p+1$ simultaneous Ehrenfeucht-Fraïssé games (see Fig. 10).

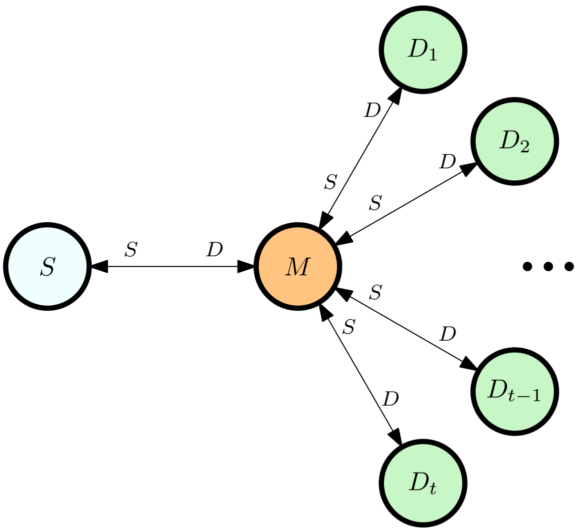

FIGURE 10. A winning strategy for $\mathrm{EF}((Y, u_1, \dots, u_p), (Y', u'_1, \dots, u'_p), n)$ using p auxiliary games $\mathrm{EF}((Y, u_i), (Y', u'_i), n+h)$.

Consider an n-step Ehrenfeucht-Fraïssé $\mathrm{EF}((\mathbf{Y}, u_1, \dots, u_p), (\mathbf{Y}', u'_1, \dots, u'_p), n)$ on $(\mathbf{Y}, u_1, \dots, u_p)$ and $(\mathbf{Y}', u'_1, \dots, u'_p)$. We build a strategy for Duplicator by considering p auxiliary Ehrenfeucht-Fraïssé games $\mathrm{EF}((\mathbf{Y}, u_i), (\mathbf{Y}', u'_i), n+h)$ on $(\mathbf{Y}, u_i)$ and $(\mathbf{Y}', u'_i)$ (for $1 \leq i \leq p$) where we play the role of Spoiler against Duplicators having a winning strategy for $n+h$ steps games.

For every vertex $v \in Y$ (resp. $v' \in Y'$) let $p(v)$ (resp. $p'(v)$) be the maximum ancestor of v (in the sense of the furthest from the root) such that $p(v) \leq u_i$ (resp. $p'(v) \leq u'_i$) for some $1 \leq i \leq p$. We partition Y and Y' as follows: for every vertex $v \in Y$ (resp. $v' \in Y'$) we put $v \in V_i$ (resp. $v' \in V'_i$) if i it the minimum integer such that $p(v) \leq u_i$ (resp. such that $p'(v) \leq u'_i$), see Fig 11.

Note that each V_i (resp. V'_i induces a connected subgraph of $\mathbf{Y}$ (resp. of $\mathbf{Y}'$).

Assume that at round $j \leq n$, Spoiler plays a vertex $v \in (\mathbf{Y}, u_1, \dots, u_p)$ (resp. a vertex $v' \in (\mathbf{Y}', u'_1, \dots, u'_p)$).

If $v \in V_i$ (resp. $v' \in V'_i$) for some $1 \leq i \leq p$ then we play v (resp. v') on $(\mathbf{Y}, u_i)$ (resp. $(\mathbf{Y}', u'_i)$). We play Duplicator on $(\mathbf{Y}', u'_1, \dots, u'_p)$ (resp. on $(\mathbf{Y}, u_1, \dots, u_p)$)

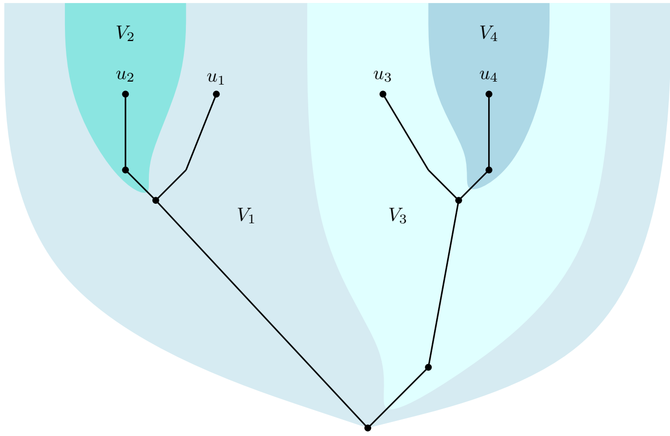

FIGURE 11. The partition (V_1, V_2, V_3, V_4) of Y induced by (u_1, u_2, u_3, u_4).

with the same move as our Duplicator opponent did on $(\mathbf{Y}', u_i)$ (resp. on $(\mathbf{Y}, u_i)$). If all the Duplicators' are not form a coherent then it is easily checked that h additional moves (at most) are sufficient for at least one of the Spoilers to win one of the p games, contradicting the hypothesis of p winning strategies for Duplicators. It follows that $(\mathbf{Y}, u_i, \dots, u_p) \equiv^n (\mathbf{Y}', u_1', \dots, u_p')$. □ □

Lemma 44. *The rooted colored forest $\mathbb{Y}_h$ (equipped with the σ-algebra Σ_h) is a relational sample space.*

Proof. Let $\varphi \in \mathrm{FO}_p$ and

$$\Omega_\varphi(\mathbb{Y}_h) = \{(v_1, \dots, v_p) \in V_h^p : \ \mathbb{Y}_h \models \varphi(v_1, \dots, v_p)\}.$$

Let $n = \mathrm{qrank}(\varphi)$. We partition V_h into equivalence classes modulo $\equiv^{n+h}$, which we denote $C_1, \dots, C_N$.

Let $i_1, \dots, i_p \in [N]$ and, for $1 \le j \le p$, let v_j and v_j' belong to C_{i_j}.

According to Lemma 43, if the heights of the lowest common ancestors of the pairs in $(v_1, \dots, v_p)$ coincide with the heights of the lowest common ancestors of the pairs in $(v_1', \dots, v_p')$ then it holds

$$(\mathbb{Y}_h, v_1, \dots, v_p) \equiv^n (\mathbb{Y}_h, v_1', \dots, v_p')$$

thus $(v_1, \dots, v_p) \in \Omega_\varphi(\mathbb{Y}_h)$ if and only if $(v_1', \dots, v_p') \in \Omega_\varphi(\mathbb{Y}_h)$.

It follows from Lemma 42 (and the definition of $\mathbb{V}_h$ and Σ_h) that each C_j is measurable. According to Lemma 42 and the encoding of the vertices of V_h, the conditions on the heights of lowest common ancestors rewrite as equalities and inequalities of coordinates. It follows that $\Omega_\varphi(\mathbb{Y}_h)$ is measurable. □ □

10.3. **The Modeling FO-limit of Residual Sequences.** We start by a formal definition of residual sequences of colored rooted trees.

Definition 19. Let $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ be a sequence of finite colored rooted trees, let N_n be the set of all sons of the root of $\mathbf{Y}_n$, and let $\mathbf{Y}_n(v)$ denote (for $v \in Y_n$) the subtree of $\mathbf{Y}_n$ rooted at v.

The sequence $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ is *residual* if

$$\limsup_{n\to\infty} \max_{v\in N_n} \frac{|\mathbf{Y}_n(v)|}{|\mathbf{Y}_n|} = 0.$$

We extend this definition to single infinite modelings.

Definition 20. A modeling colored rooted tree $\widetilde{\mathbf{Y}}$ with height at most h is *residual* if, denoting by N the neighbour set of the root, it holds

$$\sup_{v\in N} \nu_{\widetilde{\mathbf{Y}}}(\widetilde{\mathbf{Y}}(v)) = 0.$$

Note that the above definition makes sense as belonging to a same $\widetilde{\mathbf{Y}}(v)$ (for some $v \in N$) is first-order definable hence, as $\widetilde{\mathbf{Y}}$ is a relational sample space, each $\widetilde{\mathbf{Y}}(v)$ is $\Sigma_{\widetilde{\mathbf{Y}}}$-measurable.

We first prove that it is sufficient for a modeling colored rooted tree to be a modeling FO-limit of a residual sequence $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ of rooted colored trees with bounded height, it is sufficient that it is a modeling FO_1-limit of the sequence.

Lemma 45. *Assume $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ is a residual FO_1-convergent sequence of finite rooted colored trees with bounded height with residual modeling FO_1-limit $\widetilde{\mathbf{Y}}$.*

Then $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ is FO-convergent and has modeling FO-limit $\widetilde{\mathbf{Y}}$.

Proof. Let h be a bound on the height of the rooted trees $\mathbf{Y}_n$. Let $\mathbf{F}_n = I_{Y\to F}(\mathbf{Y}_n)$. Let ϖ be the formula asserting $\mathrm{dist}(x_1, x_2) \le 2h$. Then $\mathbf{F}_n \models \varpi(u, v)$ holds if and only if u and v belong to a same connected component of $\mathbf{F}_n$. According to Lemma 32, we get that $(\mathbf{F}_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$ convergent. As it is also FO_0-convergent, it is FO convergent (according to Theorem 11). As $\mathbf{Y}_n = I_{F\to Y}(\mathbf{F}_n)$, we deduce that $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ is FO-convergent.

That $\widetilde{\mathbf{Y}}$ is a modeling FO-limit of $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ then follows from Lemma 27 and Lemma 25. □ □

Now we have to relate FO_1 properties to our encoding of V_h, in order to transfer the measure μ we obtained in Theorem 7 on $S(\mathcal{B}(\mathrm{FO}_1))$ to the relational sample space formed by the connected component of $\mathbb{Y}_h$ that is an elementary limit of the considered residual sequence. To achieve this, we need some preparatory technical lemmas.

Let $\lambda^\bullet$ denote the signature obtained from λ by adding a new unary relation S (marking a *special* vertex). Let $\theta_\bullet$ be the sentence

$$(\exists x)(S(x) \wedge (\forall y\ S(y) \to (y = x)),$$

which states that a $\lambda^\bullet$ contains a unique special vertex, and let $\mathcal{I}_\bullet$ be the principal ideal of $\mathcal{B}(\mathrm{FO}_0(\lambda^\bullet))$ generated by $\neg\theta_\bullet$. Let $\mathcal{B}(\mathrm{FO}_0(\lambda^\bullet, \theta_\bullet) = \mathcal{B}(\mathrm{FO}_0(\lambda^\bullet))/\mathcal{I}_\bullet$. Then there is an obvious isomorphism of $\mathcal{B}(\mathrm{FO}_1(\lambda))$ and $\mathcal{B}(\mathrm{FO}_0(\lambda^\bullet), \theta_\bullet)$, and a corresponding homeomorphism of $S(\mathcal{B}(\mathrm{FO}_1(\lambda)))$ and $S(\mathcal{B}(\mathrm{FO}_0(\lambda^\bullet)), \theta_\bullet)$.

We consider the simple interpretation $I_\bullet$ of λ-structures in $\lambda^\bullet$-structures, which maps a $\lambda^\bullet$-structure $\mathbf{Y}$ defined as follows: let $x \simeq y$ be defined as $(x \sim y) \vee (x = y)$. Then

- the domain of $I_\bullet(\mathbf{Y})$ is defined by the formula

$$\neg\theta_\bullet \vee (\exists y_1, \dots, y_h)\Big(R(y_1) \wedge (y_h \simeq x_1) \wedge \bigwedge_{i=1}^{h-1} ((\neg S(y_i) \vee (y_i = x_1)) \wedge (y_i \simeq y_{i+1})\Big);$$

- the relation R of $I_\bullet(\mathbf{Y})$ is defined by the formula

$$\neg\theta_\bullet \wedge R(x_1) \vee \theta_\bullet \wedge S(x_1).$$

Then $I_\bullet$ maps a colored rooted tree $\mathbf{Y}$ with a single special vertex v to the colored rooted tree $\mathbf{Y}(v)$. In a sake for simplicity, we denote by $(\mathbf{Y}, v)$ (where $\mathbf{Y}$ is a λ-structure) the $\lambda^\bullet$-structure obtained by adding the new relation S with v being the unique special vertex.

Lemma 46. *For every sentence $\phi \in \mathrm{FO}_0(\lambda)$ there exists a formula $\varrho(\phi) \in \mathrm{FO}_1(\lambda)$, called* relativization *of ϕ, with the following property:*

For every colored rooted tree $\mathbf{Y}$ and every $v \in Y$ it holds

$$\mathbf{Y}(u) \models \phi \quad \iff \quad \mathbf{Y} \models \varrho(\phi)(u).$$

Proof. According to the isomorphism of $\mathcal{B}(\mathrm{FO}_1(\lambda))$ and $\mathcal{B}(\mathrm{FO}_0(\lambda^\bullet), \theta_\bullet)$, the lemma follows from the existence of $J_\bullet$ (defined by the interpretation $I_\bullet$) such that

$$\mathbf{Y}(u) \models \phi \quad \iff \quad (\mathbf{Y}, u) \models J_\bullet(\phi).$$

$\square$ $\square$

Definition 21. Let Encode : $\mathfrak{Y}_1^{(h)} \to \biguplus_{i=1}^h (\mathfrak{Y}_0^{(h)})^i$ be the mapping defined as follows: Let k be the integer such that the formula $\eta_k \in \mathrm{FO}_1(\lambda)$ stating that the height of x_1 is k belongs to T. Then $\mathrm{Encode}(T) = (T_1, \dots, T_k)$, where T_i is the set of all the sentences θ such that $\varrho(\theta) \wedge \eta_i$ belongs to T.

Lemma 47. Encode *is a homeomorphism of $\mathfrak{Y}_1^{(h)}$ and* $\mathrm{Encode}(\mathfrak{Y}_1^{(h)})$*, which is a closed subspace of $\biguplus_{i=1}^h (\mathfrak{Y}_0^{(h)})^i$.*

Proof. This lemma is a direct consequence of Lemma 42. $\square$

Definition 22. Let μ be a measure on $\mathfrak{Y}_1^{(h)}$. We define ν on $\mathbb{Y}_h$ as follows: let $\widetilde{\mu} = \mathrm{Encode}_*(\mu)$ be the push-forward of μ by Encode. For $t \in \overline{\mathbb{N}}$ we equip X_t with uniform discrete probability measure if $t < \infty$ and the Haar probability measure if $t = \infty$. For $z \in \overline{\mathbb{N}}^a$, X_z is equipped with the corresponding product measure, which we denote by λ_z.

We define the measure ν as follows: let A be a measurable subset of V_h, let $A_0 = A \cap \mathfrak{Y}_0^{(h)}$, and let $A_z = A \cap (F_z \times X_z)$. Then

$$\nu(A) = \widetilde{\mu}(A_0) + \sum_z (\widetilde{\mu} \otimes \lambda_z)(A_z).$$

(Notice that the sets A_z are measurable as $F_z \times X_z$ is measurable for every z.)

Lemma 48. *The measure μ is the push-forward of ν by the projection $P : \mathbb{Y}_h \to \mathfrak{Y}_1^{(h)}$ defined by*

$$P((T_0, T_1, \alpha_1, \dots, T_a, \alpha_a)) = \mathrm{Encode}^{-1}(T_0, \dots, T_a).$$

Proof. First notice that P is indeed continuous. Let B be a measurable set of $\mathfrak{Y}_1^{(h)}$. Let $A = P^{-1}(B)$. Then $A \cap (F_z \times X_z) = (\mathrm{Encode}(B) \cap F_z) \times X_z$ hence

$$(\widetilde{\mu} \otimes \lambda_z)(A \cap (F_z \times X_z)) = \nu(\mathrm{Encode}(B) \cap F_z)\lambda_z(X_z) = \widetilde{\mu}(\mathrm{Encode}(B) \cap F_z).$$

It follows that

$$\begin{aligned}
P_*(\nu)(B) &= \nu(A) \\
&= \widetilde{\mu}(A \cap \mathfrak{Y}_0^{(h)}) + \sum_z (\widetilde{\mu} \otimes \lambda_z)(A \cap (F_z \times X_z)) \\
&= \widetilde{\mu}(\mathrm{Encode}(B) \cap \mathfrak{Y}_0^{(h)}) + \sum_z \widetilde{\mu}(\mathrm{Encode}(B) \cap F_z) \\
&= \widetilde{\mu}\big(\mathrm{Encode}(B) \cap (\mathfrak{Y}_0^{(h)} \uplus \biguplus_z F_z)\big) \\
&= \widetilde{\mu} \circ \mathrm{Encode}(B) \\
&= \mu(B).
\end{aligned}$$

(as z ranges over a countable set and as all the F_z are measurable). Hence $\mu = P_*(\nu)$. $\square$ $\square$

Lemma 49. *Let $\mathbf{Y}_n$ be a residual* FO_1*-convergent sequence of colored rooted trees with height at most h, let μ be the limit measure of $\mu_{\mathbf{Y}_n}$ on $\mathfrak{T}_1^{(h)}$, and let $\widetilde{\mathbf{Y}}$ be the connected component of $\mathbb{Y}_h$ containing the support of ν. Then $\widetilde{\mathbf{Y}}$, equipped with the probability measure $\nu_{\widetilde{\mathbf{Y}}} = \nu$, is a modeling* FO*-limit of $(\mathbf{Y}_n)_{n\in\mathbb{N}}$.*

Proof. As $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ is elementarily convergent, the complete theory of the elementary limit of this sequence is the theory T_0 to which every point of the support of μ projects. Hence the support of μ defines a unique connected component of $\mathbb{Y}_h$. That $\widetilde{\mathbf{Y}}$ is an FO_1-modeling limit of $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ is a consequence of Lemmas 48 and 19. That it is then an FO-modeling limit of $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ follows from Lemma 45 □ □

10.4. **The Modeling FO-Limit of a Sequence of Colored Rooted Trees.** For an intuition of how the structure of a modeling FO-limit of a sequence of colored rooted trees with height at most h could look like, consider a modeling rooted colored tree $\mathbf{Y}$. Obviously, the $\mathbf{Y}$ contains two kind of vertices: the *heavy* vertices v such that the subtree $\mathbf{Y}(v)$ of $\mathbf{Y}$ rooted at v has positive $\nu_{\mathbf{Y}}$-measure and the *light* vertices for which $\mathbf{Y}(v)$ has zero $\nu_{\mathbf{Y}}$-measure. It is then immediate that heavy vertices of $\mathbf{Y}$ induce a countable rooted subtree with same root as $\mathbf{Y}$.

This suggest the following definitions.

Definition 23. A *rooted skeleton* is a countable rooted tree $\mathbf{S}$ together with a *mass function* $m : S \to (0, 1]$ such that $m(r) = 1$ (r is the root of $\mathbf{S}$) and for every non-leaf vertex $v \in S$ it holds

$$m(v) \geq \sum_{u \text{ son of } v} m(u).$$

Definition 24. Let $(\mathbf{S}, m)$ be a rooted skeleton, let S_0 be the subset of S with vertices v such that $m(v) = \sum_{u \text{ son of } v} m(u)$, let $(\mathbf{R}_v)_{v\in S\setminus S_0}$ be a countable sequence of non-empty residual λ-modeling indexed by $S \setminus S_0$, and let $(\mathbf{R}_v)_{v\in S_0}$ be a countable sequence of non empty countable colored rooted trees indexed by S_0. The *grafting* of $(\mathbf{R}_v)_{v\in S\setminus S_0}$ and $(\mathbf{R}_v)_{v\in S_0}$ on $(\mathbf{S}, m)$ is the modeling $\mathbf{Y}$ defined as follows: As a graph, $\mathbf{Y}$ is obtained by taking the disjoint union of $\mathbf{S}$ with the colored rooted trees $\mathbf{R}_v$ and then identifying $v \in S$ with the root of $\mathbf{R}_v$. The sigma algebra $\Sigma_{\mathbf{Y}}$ is defined as

$$\Sigma_{\mathbf{Y}} = \Big\{ \bigcup_{v\in S\setminus S_0} M_v \cup \bigcup_{v\in S_0} M'_v :\ M_v \in \Sigma_{\mathbf{R}_v}, M'_v \subseteq R_v \Big\}$$

and the measure $\nu_{\mathbf{Y}}(M)$ of $M \in \Sigma$ is defined by

$$\nu_{\mathbf{Y}}(M) = \sum_{v\in S\setminus S_0} \Big(m(v) - \sum_{u \text{ son of } v} m(u) \Big) \nu_{\mathbf{R}_v}(M_v),$$

where $M = \bigcup_{v\in S\setminus S_0} M_v \cup \bigcup_{v\in S} M'_v$ with $M_v \in \Sigma_{\mathbf{R}_v}$ and $M'_v \subseteq R_v$.

Lemma 50. *Let $\mathbf{Y}$ be obtained by grafting countable sequence of non-empty modeling colored rooted trees $\mathbf{R}_v$ on a rooted skeleton $(\mathbf{S}, m)$. Then $\mathbf{Y}$ is a modeling.*

Proof. We prove the statement by induction over the height of the rooted skeleton. The statement obviously holds if $\mathbf{S}$ is a single vertex rooted tree (that is if height$(\mathbf{S}) = 1$). Assume that the statement holds for rooted skeletons with height at most h, and let $(\mathbf{S}, m)$ be a rooted skeleton with height $h + 1$.

Let s_0 be the root of $\mathbf{S}$ and let $\{s_i : i \in I \subseteq \mathbb{N}\}$ be the set of the sons of s_0 in $\mathbf{S}$. For $i \in I$, $\mathbf{Y}_i = \mathbf{Y}(s_i)$ be the subtree of $\mathbf{Y}$ rooted at s_i, let $\lambda_i = \sum_{x\in Y_i} m(x)$, and let m_i be the mass function on $\mathbf{S}_i$ defined by $m_i(v) = m(v)/\lambda_i$. Also, let $\lambda_0 = 1 - \sum_{i\in I} \lambda_i$.

For each $i \in I \cup \{0\}$, if $\lambda_i = 0$ (in which case $\mathbf{R}_{s_i}$ is only assumed to be a relational sample space) we turn $\mathbf{R}_{s_i}$ into a modeling by defining a probability measure on $\mathbf{R}_{s_i}$ concentrated on s_i.

For $i \in I$, let $\mathbf{Y}_i$ be obtained by grafting the $\mathbf{R}_v$ on $(\mathbf{S}_i, m_i)$ (for $v \in S_i$), and let $\mathbf{Y}_0$ be the λ^+-modeling consisting in a rooted colored forest with single (principal) component $\mathbf{R}_{s_0}$ (that is: $\mathbf{Y}_0 = \mathsf{I}_{R \to P}(\mathbf{R}_{s_0})$). According to Lemma 18, $\mathbf{Y}_0$ is a modeling, and by induction hypothesis each $\mathbf{Y}_i$ ($i \in I$) is a modeling. According to Lemma 29, it follows that $\mathbf{F} = \amalg_{i \in I \cup \{0\}} (\mathbf{Y}_i, \lambda_i)$ is a modeling. Hence, according to Lemma 18, $\mathbf{Y} = \mathsf{I}_{F \to Y}(\mathbf{F})$ is a modeling. □ □

Our main theorem is the following.

Theorem 26. *Let $(\mathbf{Y}_n)_{n \in \mathbb{N}}$ be an* FO*-convergent sequence of finite colored rooted trees with height at most h.*

Then there exists a skeleton $(\mathbf{S}, m)$ and a family $(\mathbf{R}_v)_{v \in S}$ — where $\mathbf{R}_v$ is (isomorphic to) a connected component of $\mathbb{Y}_h$, $\Sigma_{\mathbf{R}_v}$ is the induced σ-algebra on R_v — with the property that the grafting $\mathbf{Y}$ of the $\mathbf{R}_v$ on $(\mathbf{S}, m)$ is a modeling FO*-limit of the sequence $(\mathbf{Y}_n)_{n \in \mathbb{N}}$.*

Proof. First notice that the statement obviously holds if $\lim_{n \to \infty} |Y_n| < \infty$ as then the sequence is eventually constant to a finite colored rooted tree $\mathbf{Y}$: we can let $\mathbf{S}$ be $\mathbf{Y}$ (without the colors), m be the uniform weight ($m(v) = 1/|Y|$), and $\mathbf{R}_v$ be single vertex rooted tree whose root's color is the color of v in $\mathbf{Y}$. So, we can assume that $\lim_{n \to \infty} |Y_n| = \infty$.

We prove the statement by induction over the height bound h. For $h = 1$, each $\mathbf{Y}_n$ is a single vertex colored rooted tree, and the statement obviously holds.

Assume that the statements holds for $h = h_0 - 1 \geq 1$ and let finite colored rooted trees with height at most h_0. Let $\mathbf{F}_n = \mathsf{I}_{Y \to F}(\mathbf{Y}_n)$. Then $(\mathbf{F}_n)_{n \in \mathbb{N}}$ is FO-convergent (according to Lemma 18). According to the Comb Structure Theorem, there exists countably many convergent sequences $(\mathbf{Y}_{n,i})_{n \in \mathbb{N}}$ of colored rooted trees (for $i \in I$) and an FO-convergent sequence $(\mathbf{R}_n)_{n \in \mathbb{N}}$ of special rooted forests forming a uniformly convergent family of sequences, such that $\mathsf{I}_{Y \to F}(\mathbf{Y}_n) = \mathbf{R}_n \cup \bigcup_{i \in I} \mathbf{Y}_{n,i}$.

If the limit spectrum of $(\mathsf{I}_{Y \to F}(\mathbf{Y}_n))_{n \in \mathbb{N}}$ is empty (i.e. $I = \emptyset$), the sequence $(\mathbf{Y}_n)_{n \in \mathbb{N}}$ of colored rooted trees is *residual*, and the result follows from Lemma 49.

Otherwise, let $(\lambda_i)_{i \in I}$ the limit spectrum of $(\mathsf{I}_{Y \to F}(\mathbf{Y}_n))_{n \in \mathbb{N}}$, let $\lambda_0 = 1 - \sum_{i \in I} \lambda_i$, and let $\mathbf{Y}_{n,0} = \mathsf{I}_{R \to P} \circ \mathsf{I}_{F \to Y}(\mathbf{R}_n)$. If $\lambda = 0$ then there is a connected component $\widetilde{\mathbf{Y}}_0$ of $\mathbb{Y}_h$ that is an elementary limit of $(\mathbf{Y}_{n,0})_{n \in \mathbb{N}}$; Otherwise, as $(\mathsf{I}_{F \to Y}(\mathbf{R}_n))_{n \in \mathbb{N}}$ is residual, $(\mathbf{Y}_{n,0})_{n \in \mathbb{N}}$ has, according to Lemma 49, a modeling FO-limit $\widetilde{\mathbf{Y}}_0$. By induction, each $(\mathbf{Y}_{n,i})_{n \in \mathbb{N}}$ has a modeling FO-limit $\widetilde{\mathbf{Y}}_i$. As $\mathbf{Y}_n = \mathsf{I}_{F \to Y}(\bigcup_{i \in I \cup \{0\}} \mathbf{Y}_{n,i})$, we deduce, by Corollary 3, Lemma 31, Theorem 11, and Lemma 18, that $(\mathbf{Y}_n)_{n \in \mathbb{N}}$ has modeling FO-limit $\mathsf{I}_{F \to Y}(\coprod_{i \in I \cup \{0\}} (\widetilde{\mathbf{Y}}_i, \lambda_i))$. □ □

So, in the case of colored rooted trees with bounded height, we have constructed an explicit relational sample space that allows to pullback the limit measure μ defined on the Stone space $S(\mathcal{B}(\mathrm{FO}))$.

11. Limit of Graphs with Bounded Tree-depth

Let Y be a rooted forest. The vertex x is an *ancestor* of y in Y if x belongs to the path linking y and the root of the tree of Y to which y belongs to. The *closure* Clos(Y) of a rooted forest Y is the graph with vertex set $V(Y)$ and edge set $\{\{x, y\} : x \text{ is an ancestor of } y \text{ in } Y, x \neq y\}$. The *height* of a rooted forest is the maximum number of vertices in a path having a root as an extremity. The

tree-depth $\mathrm{td}(G)$ of a graph G is the minimum height of a rooted forest Y such that $G \subseteq \mathrm{Clos(Y)}$. This notion is defined in [50] and studied in detail in [61]. In particular, graphs with bounded tree-depth serve as building blocks for *low tree-depth decompositions*, see [51, 52, 53]. It is easily checked that for each integer t the property $\mathrm{td}(G) \leq t$ is first-order definable. It follows that for each integer t there exists a first-order formula ξ with a single free variable such that for every graph G and every vertex $v \in G$ it holds:

$$G \models \xi(v) \qquad \iff \qquad \mathrm{td}(G) \leq t \text{ and } \mathrm{td}(G - v) < \mathrm{td(G)}.$$

Let $t \in \mathbb{N}$. We define the basic interpretation scheme I_t, which interprets the class of connected graphs with tree-depth at most t in the class of 2^{t-1}-colored rooted trees: given a 2^{t-1}-colored rooted tree $\mathbf{Y}$ (where colors are coded by $t-1$ unary relations $C_1, \dots, C_{t-1}$), the vertices $u, v \in Y$ are adjacent in $\mathsf{I}_t(\mathbf{Y})$ if the there is an integer i in $1, \dots, t-1$ such that $\mathbf{Y} \models C_i(v)$ and u is the ancestor of v at height i or $\mathbf{Y} \models C_i(u)$ and v is the ancestor of u at height i.

Theorem 27. *Let $(\mathbf{G}_n)_{n\in\mathbb{N}}$ be an* FO*-convergent sequence of finite colored graphs with tree-depth at most h.*

Then there exists a modeling $\mathbf{G}$ with tree-depth at most h that is a modeling FO*-limit of the sequence $(\mathbf{G}_n)_{n\in\mathbb{N}}$.*

Proof. For each $\mathbf{G}_n$, there is a colored rooted tree $\mathbf{Y}_n$ with height at most h such that $\mathbf{G}_n = \mathsf{I}_h(\mathbf{Y}_n)$. By compactness, the sequence $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ has a converging subsequence $(\mathbf{Y}_{i_n})_{n\in\mathbb{N}}$, which admits a modeling FO-limit $\mathbf{Y}$ (according to Theorem 26), and it follows from Lemma 18 that $\mathsf{I}_h(\mathbf{Y})$ is a modeling FO-limit (with tree-depth at most h) of the sequence $(\mathbf{G}_{i_n})_{n\in\mathbb{N}}$, hence a modeling FO-limit of the sequence $(\mathbf{G}_n)_{n\in\mathbb{N}}$. □ □

12. Concluding Remarks

12.1. **Selected Problems.** The theory developed here is open ended and we hope that it will encourage further researches. Here we list a sample of related problems

The first two problems concern existence of modeling FO-limits.

Problem 1. Is it true that every FO-convergent sequence of finite relational structures admit a modeling FO-limit?

In particular, it follows from Theorem 13 that there exists an FO-convergent sequence $(G_n)_{n\in\mathbb{N}}$ such that $(G_n)_{n\in\mathbb{N}}$ converges elementarily to the Rado graph and $(G_n)_{n\in\mathbb{N}}$ is L-convergent to the constant graphon 1/2. This suggests the following problem.

Problem 2. Does there exist modeling FO-limit $\mathbf{G}$ for $G(n, 1/2)$, that is a modeling such that G is elementarily equivalent to the Rado graph and for every finite labeled graph F with vertex set $\{v_1, \dots, v_p\}$ it holds

$$\nu_{\mathbf{G}}\Big(\big\{(x_1, \dots, x_p) \in G^p : v_i \mapsto x_i \text{ is isomorphism of } F \text{ and } G[x_1, \dots, x_p]\big\}\Big) = 2^{-\binom{p}{2}}?$$

Aldous-Lyons conjecture [3] states that every unimodular distribution on rooted countable graphs with bouded degree is the limit of a bounded degree graph sequence. One of the reformulations of this conjecture is that every graphing is an $\mathrm{FO}^{\mathrm{local}}$ limit of a sequence of finite graphs. The importance of this conjecture appears, for instance, in the fact that it would imply that all groups are sofic, which would prove a number of famous conjectures which are proved for sofic groups but still open for all groups.

The two next problems are related to this conjecture: the first one concerns a possible strengthening of the conjecture, and the second one is concerned with looking for an analog for rooted colored trees with bounded height.

Problem 3. Is every graphing $\mathbf{G}$ with the finite model property an FO-limit of a sequence of finite graphs?

In the case of colored rooted trees with bounded height, we have constructed an explicit relational sample space that allows to pullback the limit measure μ defined on the Stone space $S(\mathcal{B}(\mathrm{FO}))$. However, we still cannot fully characterize limits of colored rooted forests with bounded height. According to our construction, a full characterization would follow to a solution of the following problem.

Problem 4. Characterize the measures μ on $S(\mathcal{B}(\mathrm{FO}_1))$ that are limits of residual FO_1-convergent sequences of colored rooted trees with bounded height.

12.2. **Classes with bounded SC-depth.** We can generalize our main construction of limits to other tree-like classes. For example, in a similar way that we obtained a modeling FO-limit for FO-convergent sequences of graphs with bounded tree-depth, it is possible to get a modeling FO-limit for FO-convergent sequences of graphs with bounded SC-depth, where SC-depth is defined as follows [31]:

Let G be a graph and let $X \subseteq V(G)$. We denote by $\overline{G}^X$ the graph G' with vertex set $V(G)$ where $x \neq y$ are adjacent in G' if (i) either $\{x, y\} \in E(G)$ and $\{x, y\} \not\subseteq X$, or (ii) $\{x, y\} \notin E(G)$ and $\{x, y\} \subseteq X$. In other words, $\overline{G}^X$ is the graph obtained from G by complementing the edges on X.

Definition 25 (SC-depth)**.** We define inductively the class $\mathcal{SC}(n)$ as follows:

- We let $\mathcal{SC}(0) = \{K_1\}$;
- if $G_1, \dots, G_p \in \mathcal{SC}(n)$ and $H = G_1 \dot\cup \dots \dot\cup G_p$ denotes the disjoint union of the G_i, then for every subset X of vertices of H we have $\overline{H}^X \in \mathcal{SC}(n+1)$.

The *SC-depth* of G is the minimum integer n such that $G \in \mathcal{SC}(n)$.

12.3. **Classes with bounded expansion.** A graph H is a *shallow topological minor* of a graph G at depth t if some $\leq 2t$-subdivision of H is a subgraph of G. For a class $\mathcal{C}$ of graphs we denote by $\mathcal{C} \,\widetilde{\nabla}\, t$ the class of all shallow topological minors at depth t of graphs in $\mathcal{C}$. The class $\mathcal{C}$ has *bounded expansion* if, for each $t \geq 0$, the average degrees of the graphs in the class $\mathcal{C} \,\widetilde{\nabla}\, t$ is bounded, that is (denoting $\overline{\mathrm{d}}(G)$ the average degree of a graph G):

$$(\forall t \geq 0) \qquad \sup_{G \in \mathcal{C} \,\widetilde{\nabla}\, t} \overline{\mathrm{d}}(G) < \infty.$$

The notion of classes with bounded expansion were introduced by the authors in [48, 49, 51], and their properties further studied in [52, 53, 17, 18, 54, 56, 58, 59, 61, 62] and in the monograph [60]. Particularly, classes with bounded expansion include classes excluding a topological minor, like classes with bounded maximum degree, planar graphs, proper minor closed classes, etc.

Classes with bounded expansion have the characteristic property that they admit special decompositions — the so-called *low tree-depth decompositions* — related to tree-depth:

Theorem 28 ([49, 51])**.** *Let $\mathcal{C}$ be a class of graph. Then $\mathcal{C}$ has bounded expansion if and only if for every integer $p \in \mathbb{N}$ there exists $N(p) \in \mathbb{N}$ such that the vertex set of every graph $G \in \mathcal{C}$ can be partitioned into at most $N(p)$ parts in such a way that the subgraph of G induced by any $i \leq p$ parts has tree-depth at most i.*

This decomposition theorem is the core of linear-time first-order model checking algorithm proposed by Dvořák, Kráľ, and Thomas [19, 21]. In their survey on methods for algorithmic meta-theorems [32], Grohe and Kreutzer proved that (in a class with bounded expansion) it is possible eliminate a universal quantification by means of the additions of a bounded number of new relations while preserving the Gaifman graph of the structure.

By an inductive argument, we deduce that for every integer p, r and every class $\mathcal{C}$ of λ-structure with bounded expansion, there is a signature $\lambda^+ \supseteq \lambda$, such that every λ-structure $\mathbf{A} \in \mathcal{C}$ can be lifted into a λ^+-structure $\mathbf{A}^+$ with same Gaifman graph, in such a way that for every first-order formula $\phi \in \mathrm{FO}_p(\lambda)$ with quantifier rank at most r there is an existential formula $\widetilde{\phi} \in \mathrm{FO}_p(\lambda^+)$ such that for every $v_1, \ldots, v_p \in A$ it holds

$$\mathbf{A} \models \phi(v_1, \ldots, v_p) \qquad \iff \qquad \mathbf{A}^+ \models \widetilde{\phi}(v_1, \ldots, v_p).$$

Moreover, by considering a slightly stronger notion of lift if necessary, we can assume that $\widetilde{\phi}$ is a local formula. We deduce that there is an integer $q = q(\mathcal{C}, p, r)$ such that checking $\phi(v_1, \ldots, v_p)$ can be done by considering satisfaction of $\widetilde{\psi}(v_1, \ldots, v_p)$ in subgraphs induced by q color classes of a bounded coloration. Using a low-tree depth decomposition (and putting the corresponding colors in the signature λ^+), we get that there exists finitely many induced substructures $\mathbf{A}_I^+$ $(I \in \binom{[N]}{q})$ with tree-depth at most q and the property that for every first-order formula $\phi \in \mathrm{FO}_p(\lambda)$ with quantifier rank at most r there is an existential formula $\widetilde{\phi} \in \mathrm{FO}_p(\lambda^+)$ such that for every $v_1, \ldots, v_p \in A$ with set of colors $I_0 \subseteq I$ it holds

$$\mathbf{A} \models \phi(v_1, \ldots, v_p) \iff \exists I \in \binom{[N]}{q-p} : \ \mathbf{A}^+_{I \cup I_0} \models \widetilde{\phi}(v_1, \ldots, v_p).$$

Moreover, the Stone pairing $\langle \phi, \mathbf{A} \rangle$ can be computed by inclusion/exclusion from stone pairings $\langle \phi, \mathbf{A}_I^+ \rangle$ for $I \in \binom{[N]}{\leq q}$.

Thus, if we consider an FO converging-sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$, the tuple of limits of the λ^+-structures $(\mathbf{A}_n)_I^+$ behaves as a kind of approximation of the limit of the λ-structures $\mathbf{A}_n$. We believe that this presents a road map for considering more general limits of sparse graphs.

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